Useful Mathematical Facts

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1. **Symbols**
   (a) $\triangleq$ "Is defined as"
   (b) $n!$ "n-factorial"

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ (1)(2)(3)\cdots(n) & \text{if } n \neq 0 \end{cases} \tag{1}$$

   (c) Sterling's approximation

$$n! \approx \sqrt{2\pi n}(\frac{n}{e})^n \tag{2}$$

   (d) $e = 2.718281828\cdots$, the natural number
   (e) $\ln(x) = \log_e(x)$ The natural logarithm
   (f) 1_A The indicator (or characteristic) function of the set A. It tells us whether or not an element belongs to a set. It is defined as follows, $1_A : \Omega \to \{0, 1\}$.

$$1_A(\omega) = \begin{cases} 1 & \text{for all } \omega \in A \cap \Omega \\ 0 & \text{for all } \omega \in A^c \cap \Omega \end{cases} \tag{3}$$

   Another common symbol for the indicator function of the set A is ξ_A

2. **The Greek alphabet**

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

3. **Series**

$$1 + 2 + 3 + \cdots + n = \frac{(n)(n+1)}{2} \tag{4}$$

$$a^0 + a^1 + a^2 + \cdots + a^{n-1} = \frac{1 - a^n}{1 - a} \tag{5}$$

$$1 + a + a^2 + \cdots = \frac{1}{1-a}, \text{for} |a| < 1 \tag{6}$$

$$a + 2a^2 + 3a^3 + \cdots = \frac{a}{(1-a)^2}, \text{for} 0 < a < 1 \tag{7}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{8}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{9}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{10}$$

$$e^{jx} = \cos(x) + j\sin(x) \text{ where } j \triangleq \sqrt{-1} \tag{11}$$

4. **Binomial Theorem**

$$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m}\, b^m \tag{12}$$

where

$$\binom{n}{m} \triangleq \frac{n!}{(m!)\,(n-m)!} \tag{13}$$

Note from the binomial theorem it follows that

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} \tag{14}$$

5. **Exponentials**

$$a^0 = 1 \tag{15}$$

$$a^{m+n} = a^m\, a^n \tag{16}$$

$$a^{-n} = \frac{1}{a^n} \tag{17}$$

$$(ab)^n = a^n b^n \tag{18}$$

6. **Logarithms**

$$a^{(\log_a(x))} = x \tag{19}$$

$$\log_a(x\, y) = \log_a(x) + \log_a(y) \tag{20}$$

$$\log_a(x^y) = y \log_a(x) \tag{21}$$

$$\log_a(1) = 0 \tag{22}$$

$$\log_a(a) = 1 \tag{23}$$

$$\log_a(x) = (\log_b(x))/\log_b(a) \tag{24}$$

7. **Quadratic formula**
The roots of the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{25}$$

8. **Factorizations**

$$a^2 - b^2 = (a-b)(a+b) \tag{26}$$

9. **Trigonometry**

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} \tag{27}$$

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \tag{28}$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{29}$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{30}$$

$$\tan(\alpha + \beta) = \frac{\tan(x) + \tan(y)}{1 + \tan(x)\tan(y)} \tag{31}$$

10. **Hyperbolics**

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{32}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{33}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \tag{34}$$

11. **Complex Numbers**

We use the convention $j \triangleq \sqrt{-1}$. There are three ways to represent a complex number

(a) Cartesian representation

$$x = (x_r, x_i) = x_r + jx_i \tag{35}$$

where x_r and x_i are the real and imaginary components of x.

(b) Polar representation:

$$|x| \triangleq \sqrt{x_r^2 + x_i^2} \tag{36}$$

is called the magnitude of x.

$$\angle x \triangleq \text{arc tan} \frac{x_i}{x_r} \tag{37}$$

is called the phase of x.

(c) Exponential representation

$$x = |x|e^{j\angle x} = |x|(\cos(\angle x), \sin(\angle x)) \tag{38}$$

Operation on complex numbers:

(a) Addition/Substraction:

$$(x_r, x_i) + (y_r, y_i) = (x_r + y_r, x_i + y_i) \tag{39}$$

(b) Multiplication

$$|(xy)| = |x||y| \tag{40}$$
$$\angle(xy) = \angle x + \angle y \tag{41}$$

(c) Conjugation
The complex conjuage of $x = (x_r, x_i)$ is $\tilde{x} = (x_r, -x_i)$. Note $|x| = |\tilde{x}|$ and $\angle(\tilde{x}) = -\angle(x)$.

(d) Inner Product
Let $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ be complex vectors (i.e., each component of x and of y is a complex number). The inner prouct of x and y is defined as follows

$$< x, y >= x \cdot y = \sum_{i=1}^{n} x_i \tilde{y}_i \tag{42}$$

12. **Derivatives**
Let $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta_x \to 0} (f(x + \Delta x) - f(x))/\Delta x \tag{43}$$

Here are some alternative representations of the derivative

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}y = \frac{df(x)}{dx} = \frac{d}{dx}f(x) = \frac{df(u)}{du}|_{u=x} \tag{44}$$

- Exponential:

$$\frac{d}{dx}\exp(x) = \exp(x) \tag{45}$$

- Polynomial:

$$\frac{d}{dx}x^m = mx^{m-1} \tag{46}$$

- Logarithm

$$\frac{d}{dx}\ln x = \frac{1}{x} \tag{47}$$

- Sine

$$\frac{d}{dx}\sin x = \cos x \tag{48}$$

- Cosine

$$\frac{d}{dx}\cos x = -\ sinx \tag{49}$$

- Linear combinations

$$\frac{d}{dx}((a)f(x) + (b)g(x)) = (a)\frac{d}{dx}f(x) + (b)\frac{d}{dx}g(x) \tag{50}$$

- Products

$$\frac{df(x)g(x)}{dx} = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx} \tag{51}$$

- Chain Rule
Let $y = f(x)$ and $z = g(y)$

$$\frac{dy}{dx} = \frac{dz}{dy}\frac{dy}{dx} \tag{52}$$

You can think of the chain rule in the following way: x changes y which changes z. How much z changes when x changes is the product of how much y changes when x changes times how much z changes when y changes. Here is a simple example that uses the chain rule

$$\frac{d\exp(ax)}{dx} = \frac{d\exp(ax)}{dax}\frac{ax}{dx} = \exp(ax)(a) \tag{53}$$

13. **Indefinite Integrals**
The **indefinite integral** of the function f is a function whose derivative is f (i.e., the antiderivative of is f). This function is unique up to addition of arbitrary constant. The expression

$$\int f(x)dx = F(x) + C \tag{54}$$

means that $F'(x) = f(x)$. The C reminds us that the derivative of $F(x)$ plus any arbitrary constant is also $f(x)$.

- Linear Combinations

$$\int af(x) + bg(x)dx = a\int f(x)dx + b\int g(x)dx \tag{55}$$

- Polynomials

$$\int x^m dx = \frac{x^{m+1}}{m+1} + C \tag{56}$$

- Exponentials

$$\int \exp(x)dx = \exp(x) + C \tag{57}$$

- Logarithms

$$\int \frac{1}{x}dx = \ln(x) + C \tag{58}$$

- Integration by parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \tag{59}$$

The formula for integration by parts easily follows from the formula for the derivative of the product of $f(x)g(x)$.

1 Continuity

- Uniform continuity of functions: Let A, B be metric spaces. A function $f : A \to B$ is uniformly continuous if for every $\epsilon > 0$ there is $\delta > 0$ such that for all x_i, x_j such taht $d(x_i, x_j) < \delta$ it follows that $d(f(x)i), f(x_j)) < \delta$.
  Every uniformly continuous function is continuous but not every continuous function is uniformly continuous. For example $f(x) = 1/x$ is continuous but not uniformly continuous. If A is a compact metric space then every continuous function $f : M \to N$ is uniformly continuous. If (x_k) is a Cauchy sequence and f is uniformly continuous then $(f(x_n)$ is a Cauchy sequence.
- Absolute Continuity of Functions. A function f is absolutely conitnuous if for every $\epsilon > 0$ there is $\delta > 0$ such that for any sequence of disjoint intervals $[x_k, y_k]$, $k = 1, \cdots, n$ that satisfies

$$\sum_{k=1}^{n}(y_k - x_k) < \delta \tag{60}$$

then

$$\sum_{k=1}^{n}|f(y_k) - f(x_k)| < \epsilon \tag{61}$$

Absolute continuity implies uniform continuity and therefore continuity. Lipschitz continuity implies absolute contiuity.

- Absolute Continuity of Measures. Let$P$ and Q are measures on the same space. P is absolutely continues withr respect to Q if $Q(A) = 0 \to P(A) = 0$, we write it $P << Q$. Radon-Nikodym showed that if P is absolutely continuous with respect to Q then P has density f (Radon-Nikodym derivative) with respect to Q. This is a measurable function such that for any measurable set A

$$P(A) = \int_A f dQ \tag{62}$$

2 Random Variables

- Beta Variables:

$$R \sim \text{Beta}(\beta_1, \beta_2) \tag{63}$$

$$p(r) = \text{Beta}(r, \beta_1, \beta_2) = \frac{\Gamma(\beta_1 + \beta_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} (r)^{\beta_1 - 1}(1 - r)^{\beta_2 - 1} \tag{64}$$

$$\text{E}(R) = \frac{\beta_1}{\beta_1 + \beta_2} \tag{65}$$

$$\text{Var}(R) = \frac{\beta_1 \beta_2}{(\beta_1 + \beta_2)^2(\beta_1 + \beta_2 + 1)} \tag{66}$$

The following formula provides the parameters of the Beta distribution that match a desired mean m and variance s^2

$$\beta_1 = \frac{1 - m - c^2 m}{c^2} \tag{67}$$

$$\beta_2 = \frac{1 - m}{m} \beta_1 \tag{68}$$

$$\stackrel{\text{def}}{=} \frac{s}{m} \tag{69}$$

- Gamma Function:

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du, \quad \text{for } x > 0 \tag{70}$$

The gamma function has the following properties

$$\Gamma(x + 1) = x\Gamma(x) \tag{71}$$

$$\Gamma(x) = (n - 1)!, \quad \text{for } n = 1, 2 \cdots \tag{72}$$

- Logistic Function:

$$\text{logistic}(x) = \frac{1}{1 + e^{-x}} \tag{73}$$

3 History

- The first version of this document was written by Javier R. Movellan in 1994. The document was 6 pages long.
- The document was made open source under the GNU Free Documentation License Version 1.1 on August 9 2002, as part of the Kolmogorov project.
- October 9, 2003. Javier R. Movellan changed the license to GFDL 1.2 and included an endorsement section.