Calculus of Variations

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Our goal is to find a function x that minimizes the following function

$$I(x) = \int_a^b F(t, x(t), \dot{x}(t))dt \tag{1}$$

where

$$\dot{x}(t) \stackrel{\text{def}}{=} \frac{dx(t)}{dt} \tag{2}$$

We will find a necessary condition for x to be at a minimum. The condition is equivalent to the zero gradient condition for the discrete time case.

Let x be a local minimum of F. Let now define a family of functions of the following form

$$h(t) = x(t) + \epsilon\delta(t) \tag{3}$$

where δ is an arbitrary xunction with continuous second partial derivatives and such that $\delta(a) = \delta(b) = 0$. Let

$$\tilde{I}(\epsilon) = I((h(\cdot, \epsilon)) = \int_a^b F(t, h(t), \dot{h}(t))dt \tag{4}$$

A necessary condition for x to minimize I is that

$$\left.\frac{d\tilde{I}(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0 \tag{5}$$

thus

$$\left.\int_a^b \left(\frac{dF(t, h(t), \dot{h}(t))}{dh(t)}\delta(t) + \frac{dF(t, h(t), \dot{h}(t))}{d\dot{h}(t)} \frac{d\dot{h}(t)}{d\epsilon} \right) dt\right|_{\epsilon=0} = 0 \tag{6}$$

Note for a fixed f

$$\frac{d\dot{h}(t)}{d\epsilon} = \epsilon\dot{\delta}(t) \tag{7}$$

for $\epsilon = 0$

$$\frac{dF(t, h(t), \dot{h}(t))}{dh(t)} = \frac{dF(t, x(t), \dot{x}(t))}{dx(t)} \tag{8}$$

$$\frac{dF(t, h(t), \dot{h}(t))}{d\dot{h}(t)} = \frac{dF(t, x(t), \dot{x}(t))}{d\dot{x}(t)} \tag{9}$$

Moreover, using integration by parts

$$\int_a^b \frac{dF(t, x(t), \dot{x}(t))}{d\dot{x}(t)}\dot{\delta}(t)dt = F(t, x(t), \dot{x}(t))\delta(t)|_a^b \tag{10}$$

$$- \int_a^b \frac{d}{dt}\left(\frac{dF(t, x(t), \dot{x}(t)}{d\dot{x}(t)}\right)\delta(t)dt \tag{11}$$

$$= - \int_a^b \frac{d}{dt}\left(\frac{dF(t, x(t), \dot{x}(t)}{d\dot{x}(t)}\right)\delta(t)dt \tag{12}$$

because $\delta(a) = \delta(b) = 0$. Thus

$$\int_a^b \left(\frac{dF(t, x(t), \dot{x}(t))}{dx(t)} - \frac{d}{dt}\left(\frac{dF(t, x(t), \dot{x}(t))}{d\dot{x}(t)}\right)\right)\delta(t)dt \tag{13}$$

The fundamental lemma of calculus of variations states that if

$$\int_a^b m(x)g(x)dx = 0 \tag{14}$$

for all g with continuous second partial derivatives, then

$$m(x) = 0, \text{ for } x \in (a,b) \tag{15}$$

Applying the Lemma to our case, we get that

$$\frac{dF(t,x(t),\dot{x}(t))}{dx(t)} - \frac{d}{dt}\left(\frac{dF(t,x(t),\dot{x}(t))}{d\dot{x}(t)}\right) = 0, \quad \text{for } t \in (a,b) \tag{16}$$

Or in more succinct notation

$$\frac{dF}{dx} - \frac{d}{dt}\left(\frac{dF}{d\dot{x}}\right) = 0, \tag{17}$$

This is called the Euler-Lagrange differential equation.

1 Examples

1.1 Shortest path between two points on a plane

Let x be a function in the real plane. Points of this function take the form $t, x(t)$. We constrain the function to start at the origin $(0, a)$ and end at (T, b). We can sample the function at a set of of points $(x_1, t_1), (x_2, t_2) \cdots$ on the plane, The total length of the resulting trajectory is

$$I(x) = \sum_i \sqrt{\delta_{t,i}^2 + \delta_{x,i}^2} = \sum_i \sqrt{\delta_{t,i}^2 + (\delta_{t,i}\frac{\delta_{x,i}}{\delta_{t,i}})^2} = \sqrt{1 + \left(\frac{\delta_{x,i}}{\delta_{t,i}}\right)^2}\delta_{t,i} \tag{18}$$

where $\delta_{x,i} = x_{i+1} - x_i$ and $\delta_{t,i} = t_{i+1} - t_i$. Thus, in the limit

$$I(x) = \int_a^b F(t, x(t), \dot{x}(t))\ dt \tag{19}$$

$$F(t, x(t), \dot{x}(t)) = \sqrt{1 + (\dot{x}(t))^2} \tag{20}$$

$$\tag{21}$$

In this case F does only depend on $\dot{x}$, thus the Euler-Lagrange equation takes the following form

$$0 = -\frac{d}{dt}\Big(\frac{d}{d\dot{x}(t)}\sqrt{1 + (\dot{x}(t))^2}\Big) = -\frac{d}{dt}\frac{\dot{x}(t)}{\sqrt{1 + (\dot{x}(t))^2}} \tag{22}$$

$$= -\frac{\ddot{x}(t)}{\sqrt{1 + (\dot{x}(t))^2}} + \frac{\dot{x}^2(t)\ddot{x}(t)}{(1 + (\dot{x}(t))^2)^{3/2}} \tag{23}$$

$$= \frac{\dot{x}^2(t)\ddot{x}(t) - (1 + (\dot{x}(t))^2)\ddot{x}(t)}{(1 + (\dot{x}(t))^2)^{3/2}} = \frac{\ddot{x}(t)}{(1 + (\dot{x}(t))^2)^{3/2}} \tag{24}$$

Thus a straight line between $(0, a)$ and (T, b) would satisfy the Euler-Lagrange equation.

2 Appendix

Lemma 1. *Fundamental Lemma of Calculus of Variations*

If

$$\int_a^b m(x)g(x)dx = 0 \tag{25}$$

for all g with continuous second partial derivatives, and $g(a) = g(b) = 0$ then

$$m(x) = 0, \text{ for } x \in (a, b) \tag{26}$$

Proof. Let

$$g(x) = m(x)(x - a)(b - x) \tag{27}$$

Note g is continuous and $m(x)g(x) \geq 0$ for $x \in [a, b]$. If the integral of a non-negative function is zero then the function must be zero at all points, i.e., $m(x)g(x) = 0$ for $x \in [a, b]$. Thus

$$m(x)^2(x - a)(b - x) = 0, \text{ for } x \in [a, b] \tag{28}$$

and since $(x - a)(b - x) \geq 0$ for $x \in [a, b]$ then $m(x)^2 \geq 0$ for $x \in [a, b]$, thus $m(x) = 0$ for $x \in [a, b]$. □

References