Physics for Robotics and Animation:
A Gaussian Approach

(draft under construction, poorly debugged)

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Many problems in robotics and computer animation involve motion of constrained systems. For example, we may want to know what happens when we apply forces in different parts of a robot arm. In order to apply the Newtonian equations of motion we would need to know not only the forces we are applying to the arm but also the forces imposed by the robot arm constraints. In general describing the motion constraints of a system is relatively easy, but describing the forces that enforce such constraints may be tedious. Lagrange and Hamilton found ways to solve this problem using the calculus of variations. Less known however is the fact that Gauss found a different approach that framed constrained motion as a least squares problem, thus solvable using standard linear regression methods [**?**]. In this document we introduce the Gaussian approach to constrained motion and show how it applies to classic robotics and computer animation problems.

1 Single Particle Systems

Notation:

- By default we leave time dependencies implicit. For example

  $$x = h(\theta) \tag{1}$$

  is shorthand notation for

  $$x_t = h(\theta_t), \text{ for } t \in [0, \infty) \tag{2}$$

  The advantage of this shorthand notation is that it is compact and easy to read. The disadvantage is that one needs to keep track of what symbols represents entities that are a function of time and what symbols represent constants.

- For the standard reference frame: The first axis is the horizontal axis, the second axis the vertical axis and the third axis is the depth axis.

Here we analyze the Newtonian dynamics of a particle with mass m concentrated on a single point and subject to kinematic constrains

$$x = h(\theta) \tag{3}$$

where $x \in \mathcal{R}^3$ is a vector with the particle's coordinates and $h(\cdot)$ is a function representing the kinematic constraints, the particle's trajectory in Euclidean space has to adhere to the function h.. We refer to h as the *kinematic constraints*. For example, the particle mass may be attached to a device that forces it to rotate around a circle of radius r centered at the origin, i.e.

$$x = h(\theta) = \begin{bmatrix} r\cos(\theta) \\ r\sin\theta \\ 0 \end{bmatrix} \tag{4}$$

In this case θ is a scalar but in general it could be a vector.

Let f represent the external forces applied to the point mass. In addition the particle is subject to some internal force e that enforces the kinematic constraints. We let $\hat{f}$ represent the effective forces, i,e., the sum of the forces applied to the system, plus the internal constrain forces. Applying Newton's law we get

$$\hat{f} = f + e = m\ddot{x} \tag{5}$$

Note

$$\dot{x} = \frac{dx}{dt} = J\dot{\theta} \tag{6}$$

where

$$J = \frac{\partial h(\theta)}{\partial \theta'} \tag{7}$$

is the Jacobian of h with respect to θ. and

$$\dot{\theta} = \frac{d\theta}{dt} \tag{8}$$

Using the product rule of derivatives we get

$$\ddot{x} = \frac{d^2x}{dt^2} = \dot{J}\dot{\theta} + J\ddot{\theta} \tag{9}$$

where

$$\dot{J} = \frac{d}{dt}J = \sum_i \dot{\theta}_i \frac{\partial J}{\partial \theta_i} \tag{10}$$

$$\ddot{\theta} = \frac{d^2\theta}{dt^2} \tag{11}$$

Thus, in θ coordinates Newton's equation (5) takes the following form

$$\boxed{mJ\ddot{\theta} = f - m\dot{J}\dot{\theta} + e} \tag{12}$$

Typically we know $m, h, f, J, \theta, \dot{J}, \dot{\theta}$ and our goal is to get the acceleration $\ddot{\theta}$. Note (12) tells us that the vector $f - m\dot{J}\dot{\theta}$ is a linear combination of the columns of mJ.vector.

The problem with the Newtonian equation of motion as described in (12) is that in addition to the external force f we apply to the system it requires for us to know the internal force e that enforce the system's motion constraints. If we knew f and e we could easily obtain the acceleration $\ddot{\theta}$. Unfortunately, in general we don't know e. Gauss, showed that if the system adheres to Newton's laws then the internal forces $\hat{e}$ are the smallest forces that are consistent with the equations of motion. In particular

$$\hat{e} = \operatorname*{argmin}_e \frac{1}{m}\|e\|^2 = \operatorname*{argmin}_e \sum_{i=1}^n e_i^2 \tag{13}$$

where $\hat{e}$ represents the constraint forces, computed using the principle of least constraint. In Section 2.1 we will provide an intuitive justification for Gauss' principle of least constraint. For now we will simply study the mechanics of how to use Gauss' approach. Gauss' principle implies that (12) can be solved using the standard weighted least squares method (see Appendix Section 8.1). Thus

$$\begin{aligned} \ddot{\theta} &= \Big((mJ)'\frac{1}{m}(mJ)\Big)^{-1}(mJ)'\frac{1}{m}(f - m\dot{J}\dot{\theta}) \\ &= (J'mJ)^{-1}J'(f - m\dot{J}\dot{\theta}) \end{aligned} \tag{14}$$

More succinctly, Gauss's least squares equation of motion looks as follows

$$\boxed{M\ddot{\theta} + C\dot{\theta} = \tau} \tag{15}$$

where

$$M = mJ'J \tag{16}$$

is the **mass** [1] with respect to θ.

$$C = mJ'\dot{J} \tag{17}$$

is the **apparent force** [2] on θ, and

$$\tau = J'f \tag{18}$$

is the **force**[3] on θ.

Note since the force e is the residual of the least squares problem in (12) it is orthogonal to the columns of mJ. Moreover since the velocity

$$\dot{x} = J\dot{\theta} \tag{19}$$

is in the column space of J it follows that $e \cdot \dot{x} = 0$. Thus the force e produces no work, i.e.,

$$W_t = \int_0^t e_\tau \cdot x_\tau \, d\tau = 0 \tag{20}$$

Note the difference between the original Newtonian equation (**??**) and the Gaussian version (15) is that the later on does not require knowledge of the internal constraint forces. We just need to know the kinematic constrain equation h, the state of the system $\theta, \dot{\theta}$ and the external force f we apply to the system.

1.1 Computer Simulations

In most computer simulations we know the current state θ_t the current velocity $\dot{\theta}_t$, the constrain equation h and the external force f. We can then use Gauss' equation to compute the acceleration

$$\ddot{\theta} = M^{-1}\Big(\tau - C\dot{\theta}\Big) \tag{21}$$

We then take a small time step δ and compute the state at $t + \delta$

$$\dot{\theta}_{t+\delta} = \dot{\theta} + \delta\ddot{\theta} \tag{22}$$

$$\theta_{t+\delta} = \theta + \delta\dot{\theta} + \frac{\delta^2}{2}\ddot{\theta} \tag{23}$$

$$x_{t+\delta} = h(\theta_{t+\delta}) \tag{24}$$

[1] Also known as the mass matrix, inertia matrix, inertia tensor, and moment of inertia.
[2] Also known as fictitious force matrix, or centrifugal/Coriolis force matrix.
[3] Also known as generalized force, or torque.

1.2 Example: Rigid Pendulum

Consider a point particle of mass m rigidly attached to a massless rod (See Figure 1) so that its motion adheres to the following kinematic constraint equation

$$x = h(\theta) = r \begin{bmatrix} \sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \tag{25}$$

Note

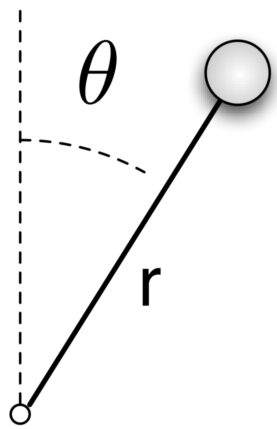

Figure 1: *Rigid Pendulum.*

$$J = r \begin{bmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{bmatrix} \tag{26}$$

$$\dot{J} = \dot{\theta}\frac{\partial J}{\partial \theta} = -\dot{\theta} r \begin{bmatrix} \sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \tag{27}$$

$$\tag{28}$$

and

$$M = mJ'J = mr^2 \tag{29}$$

$$C = mJ'\dot{J} = 0 \tag{30}$$

$$\tau = J'f = J' \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} = mgr\sin\theta \tag{31}$$

where g is the gravitational acceleration constant. Thus Gauss' equation of motion

$$M\ddot{\theta} + C\dot{\theta} = \tau \tag{32}$$

takes the following form

$$mr^2\ddot{\theta} = mrg\sin\theta \tag{33}$$

Equivalently

$$\ddot{\theta} = \frac{g\sin\theta}{r} \tag{34}$$

It is illuminating to compute the constrain forces. Using (5)

$$e = f - m\dot{J}\dot{\theta} - mJ\ddot{\theta} \tag{35}$$

$$= \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} - mr \begin{bmatrix} \sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \dot{\theta} - mr \begin{bmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{bmatrix} \frac{g}{r}\sin\theta \tag{36}$$

$$= \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} - mx\dot{\theta} + mg \begin{bmatrix} \cos\theta\sin\theta \\ \sin^2\theta \\ 0 \end{bmatrix} \tag{37}$$

$$= -mx\dot{\theta} + mgx\cos\theta = mx(g\cos\theta - \dot{\theta}) \tag{38}$$

Note the force e is parallel to x. If the angular velocity $\dot{\theta}$ is larger than $g\cos\theta$ then it points towards the origin, to counteract the centrifugal force. If $\dot{\theta}$ is smaller than $g\cos\theta$ it points away from the origin, to counteract the component of the gravity pointing towards the origin. Note, as expected, the force e is orthogonal to the velocity

$$\dot{x} = J\dot{\theta} = \begin{bmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{bmatrix} \dot{\theta} \tag{39}$$

and thus it produces no work.

1.3 Example: Spring Pendulum

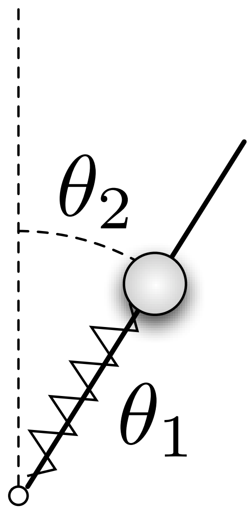

Figure 2: *Spring Pendulum.*

Imagine a ball of mass m attached to a spring rotating vertically around the z (depth) axis. A rod passes through a hole of the spring so it can only move radially (see Figure 2). Let θ_1 represent the length of the spring and θ_2 the rotation angle. We have

$$x = \theta_1 \begin{bmatrix} \cos\theta_2 \\ \sin\theta_2 \\ 0 \end{bmatrix} \tag{40}$$

Thus

$$J = \frac{\partial x}{\partial \theta'} = \begin{bmatrix} \cos\theta_2 & -\theta_1 \sin\theta_2 \\ \sin\theta_2 & \theta_1 \cos\theta_2 \\ 0 & 0 \end{bmatrix} \tag{41}$$

and

$$M = mJ'J = m \begin{bmatrix} 1 & 0 \\ 0 & \theta_1^2 \end{bmatrix} \tag{42}$$

Moreover

$$\dot{J} = \sum_{i=1}^{2} \dot{\theta}_i \frac{\partial J}{\partial \theta_i} \tag{43}$$

$$= \dot{\theta}_1 \begin{bmatrix} 0 & -\sin\theta_2 \\ 0 & \cos\theta_2 \\ 0 & 0 \end{bmatrix} + \dot{\theta}_2 \begin{bmatrix} -\sin\theta_2 & -\theta_1\cos\theta_2 \\ \cos\theta_2 & -\theta_1\sin\theta_2 \\ 0 & 0 \end{bmatrix} \tag{44}$$

Thus

$$C = mJ'\dot{J} = m\sum_i \dot{\theta}_i J' \frac{\partial J}{\partial \theta_i} \tag{45}$$

$$= m\dot{\theta}_1 \begin{bmatrix} 0 & 0 \\ 0 & \theta_1 \end{bmatrix} + m\dot{\theta}_2 \begin{bmatrix} 0 & -\theta_1 \\ \theta_1 & 0 \end{bmatrix} \tag{46}$$

$$= m \begin{bmatrix} 0 & -\theta_1\dot{\theta}_2 \\ \theta_1\dot{\theta}_2 & \theta_1\dot{\theta}_1 \end{bmatrix} \tag{47}$$

Regarding the torques, we apply a rotational torque to produce angular acceleration

$$\begin{bmatrix} 0 \\ \tau_r \end{bmatrix} \tag{48}$$

The spring produces a force directly on θ_1,

$$\begin{bmatrix} -\kappa_3\theta_1 \\ 0 \end{bmatrix} \tag{49}$$

We can also have rotational friction and radial friction

$$\begin{bmatrix} -\kappa_1\dot{\theta}_1 \\ -\kappa_2\dot{\theta}_2 \end{bmatrix} \tag{50}$$

Finally gravitational force is applied on x, and thus needs to be converted into a generalized force

$$J' \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} = \begin{bmatrix} -mg\sin\theta_2 \\ -mg\theta_1\cos\theta_2 \end{bmatrix} \tag{51}$$

Thus Gauss' equations of

$$\tau = M\ddot{\theta} + C\dot{\theta} \tag{52}$$

take the following form

$$\begin{bmatrix} -\kappa_3\theta_1 - \kappa_1\dot{\theta}_1 - mg\sin\theta_2 \\ \tau_r - \kappa_2\dot{\theta}_2 - mg\theta_1\cos\theta_2 \end{bmatrix} = m\begin{bmatrix} 1 & 0 \\ 0 & \theta_1^2 \end{bmatrix}\ddot{\theta} + m\begin{bmatrix} 0 & -\theta_1\dot{\theta}_2 \\ \theta_1\dot{\theta}_2 & \theta_1\dot{\theta}_1 \end{bmatrix}\dot{\theta} \tag{53}$$

2 Multi Particle Systems

Consider a system made of n particles with mass $m_1, \cdots m_n$. Let $f_i \in \mathcal{R}^3$ the force applied to a particle with position $x_i \in \mathcal{R}^3$.

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \tag{54}$$

Let and m a diagonal matrix with $3 \times n$ diagonal elements

$$m = \begin{bmatrix} m_1 & & & & & & \\ & m_1 & & & & & \\ & & m_1 & & & & \\ & & & \ddots & & & \\ & & & & m_n & & \\ & & & & & m_n & \\ & & & & & & m_n \end{bmatrix} \tag{55}$$

The position of the particles are constrained to follow the constraint function h, i.e., $x = h(\theta)$, or equivalently

$$x_i = h_i(\theta), \text{for } i = 1, \cdots n \tag{56}$$

Newton's law tells us that

$$\hat{f} = f + e = m\ddot{x} \tag{57}$$

where e represents the unknown forces asserted by the kinematic constrains s h. Note

$$\dot{x} = J\dot{\theta} \tag{58}$$

where J is the Jacobian of the entire set of particles

$$J = \frac{\partial x}{\partial \theta'} = \begin{bmatrix} J_1 \\ \vdots \\ J_n \end{bmatrix}, \tag{59}$$

and J_i the Jacobian for particle i

$$J_i = \frac{\partial h_i(\theta)}{\partial \theta'} \tag{60}$$

Moreover, using the product rule of derivatives

$$\ddot{x} = \dot{J}\dot{\theta} + J\ddot{\theta} \tag{61}$$

where

$$\dot{J} = \frac{dJ}{dt} = \begin{bmatrix} \dot{J}_1 \\ \vdots \\ \dot{J}_n \end{bmatrix}, \tag{62}$$

where

$$\dot{J}_i = \frac{dJ_i}{dt} = \sum_{j=1}^{d} \dot{\theta}_j \frac{\partial J_i}{\partial \theta_j} \tag{63}$$

Thus, in θ coordinates Newton's equation (57) takes the following form

$$\boxed{mJ\ddot{\theta} = f - m\dot{J}\dot{\theta} + e} \tag{64}$$

Typically we know $m, h, f, J, \theta, \dot{J}, \dot{\theta}$ and our goal is to get the acceleration $\ddot{\theta}$. Note (64) tells us that the vector $f - m\dot{J}\dot{\theta}$ is a linear combination of the columns of mJ. The unknown force e is the residual vector.

If we knew the constraint forces e we could obtain the acceleration $\ddot{\theta}$. Unfortunately, in general we don't know e. Gauss showed that if the system adheres to Newton's law then the actual constraint forces $\hat{e}$ are the smallest possible forces consistent with (64). In particular

$$\hat{e} = \operatorname*{argmin}_e e' m^{-1} e = \sum_i \frac{\|e_i\|^2}{m_i} \tag{65}$$

Thus (64) can be solved using the standard weighted least squares method (see Appendix), i.e.

$$\begin{aligned}\ddot{\theta} &= ((mJ)'m^{-1}(mJ))^{-1}mJm^{-1}(f - m\dot{J}\dot{\theta}) \\ &= (J'mJ)^{-1}J'(f - m\dot{J}\dot{\theta}) \end{aligned} \tag{66}$$

More succinctly

$$\boxed{M\ddot{\theta} = \tau - C\dot{\theta}} \tag{67}$$

where

$$M = J'mJ = \sum_{i=1}^{n} M_i \tag{68}$$

$$M_i = m_i J_i' J \tag{69}$$

is the **mass**[4] with respect to θ.

$$C = mJ'\dot{J} = \sum_{i=1}^{n} C_i \tag{70}$$

$$C_i = m_i J_i' \dot{J}_i \tag{71}$$

is the **apparent force matrix** [5] on θ, and

$$\tau = J'f = \sum_{i=1}^{n} \tau_i \tag{72}$$

$$\tau_i = J' f_i \tag{73}$$

is the **total force**[6] on θ.

2.1 Gauss' Principle of Least Constraint

Gauss [**?**] defined the principle of least constraint as follows

[4]Also known as mass matrix, inertia matrix, inertia tensor, and moment of inertia.
[5]Also known as the fictitious force matrix, or centrifugal/Coriolis force matrix.
[6]Also known as generalized force, or torque.

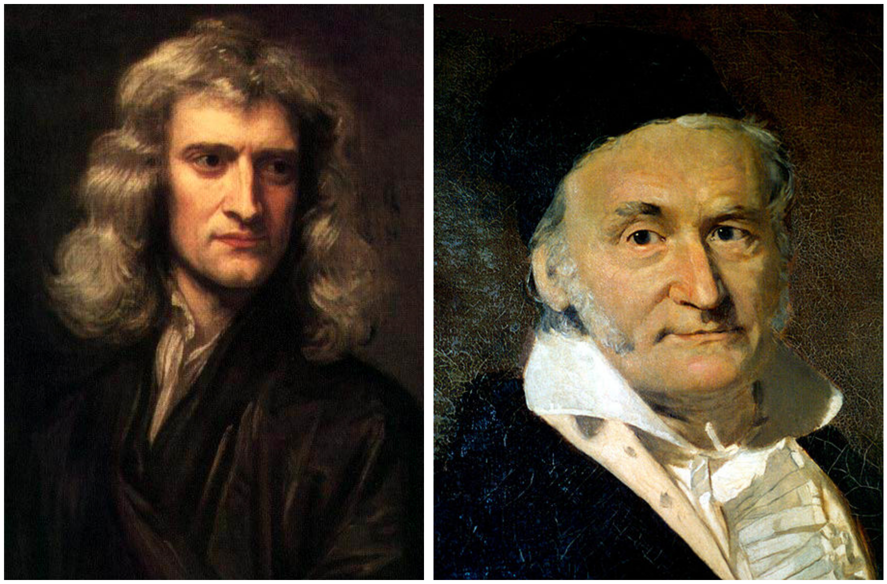

Figure 3: **Left:** Sir Isaac Newton (1643-1727). **Right** Johann Carl Friedrich Gauss (1777-1855)

> *"The motion of a system of material points, constrained in an arbitrary manner, and subjected to arbitrary forces at any moment of time, takes place in a manner which is as similar as possible to the motion that would be performed by these points if they were free, i.e. with least-possible forcing — the measure of forcing during the time dt being defined as the sum of the products of the mass of each point and the square of the distance of the point from the position which it would occupy if it were free."*

Note that given a small time period Δt the displacement of a completely free particle i if we apply the force f_i to it would be as follows

$$\tilde{\Delta} x_i = \dot{x}_i \Delta t + \frac{1}{2} m_i^{-1} f_i \Delta t^2 \tag{74}$$

If in addition to f we apply the constrain force e_i the displacement would be as follows

$$\Delta x_i = \dot{x}_i \Delta t + \frac{1}{2} m_i^{-1} (f_i + e_i) \Delta t^2 \tag{75}$$

The difference between the two displacements is as follows

$$\delta x_i = \Delta x_i - \tilde{\Delta} x_i = \frac{1}{2} m^{-1} e_i \Delta t^2 \tag{76}$$

Gauss principle tells us that the actual constraining forces would minimize the squared difference in displacements across all the particles, weighted by their masses

$$\hat{e} = \underset{e}{\operatorname{argmin}} \sum_i m_i \|\delta x_i\|^2 = \underset{e}{\operatorname{argmin}} \sum_i \frac{\|e_i\|^2}{m_i} = \underset{e}{\operatorname{argmin}}\, e' m^{-1} e \tag{77}$$

Thus minimizing the displacement is equivalent to minimizing the "size" of the constraint forces (thus principle of least constraint), where size is defined with respect to the norm

$$\|e\|_m^2 = e' m^{-1} e \tag{78}$$

Gauss' principle may be derived from the requirement that the constrain forces produce no work. Note given a period of time Δt the total work produced by the constrain forces is as follows

$$\Delta W_e = \sum_{i=1}^n e_i \cdot \Delta x_i = \sum_{i=1}^n e_i \cdot \Delta x_i \tag{79}$$

$$= \sum_{i=1}^n e_i \cdot \left(\delta x_i \Delta_t + \frac{1}{2} \ddot{x}_i \Delta t^2 \right) \tag{80}$$

$$= \Delta t \sum_{i=1}^n m_i^{-1} e_i \cdot m_i J_i \dot{\theta}_i + \frac{1}{2} \Delta t^2 \sum_{i=1}^n e_i \cdot m_i^{-1} \hat{f}_i \tag{81}$$

$$= \Delta t < e, mJ\dot{\theta} >_m + \frac{1}{2} \Delta t^2 < e, \hat{f} >_m \tag{82}$$

where the inner product $<, >_m$ is defined as follows

$$< u, v >_m = u' m^{-1} v \tag{83}$$

We note that the weighted least squares solution e is orthogonal[7] to the space spanned by the columns of mJ (see Appendix) thus, it is orthogonal to $mJ\dot{\theta}$, i.e.

$$< e, mJ\dot{\theta} >_m = < e, \hat{f} >_m = 0 \tag{84}$$

[7] Here orthogonality is defined with respect to the $<, >_m$ inner product.

Thus $\Delta W_e = 0$, in other words, Gauss's minimum constraint principle guarantees that the constrain forces produce no work, and therefore have no effect on the total kinetic energy of the particles.

2.2 Example: Two Linked Particles

Consider two particles moving along a rod in the x axis and linked so that they are separated by a constant distance c. Thus the motion constraint equation is as follows

$$x_1 = h_1(\theta) = \begin{bmatrix} \theta \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = h_2(\theta) = \begin{bmatrix} \theta + c \\ 0 \\ 0 \end{bmatrix} \tag{85}$$

Thus the Jacobian and Jacobian velocities are as follows

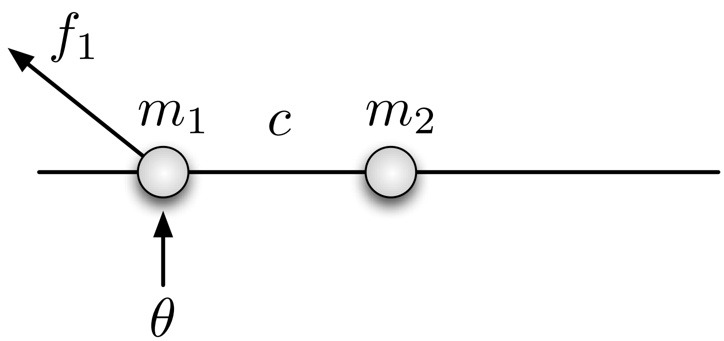

Figure 4: *Two linked particles.*

$$J_1 = J_2 = \begin{bmatrix} \theta + c \\ 0 \\ 0 \end{bmatrix} \tag{86}$$

$$\dot{J}_1 = \dot{J}_2 = \mathbf{0} \tag{87}$$

Thus

$$M_1 = m_1 J_1' J_1 = m_1 \tag{88}$$

$$M_2 = m_2 J_2' J_2 = m_2 \tag{89}$$

$$M = M_1 + M_2 = m_1 + m_2 \tag{90}$$

and

$$C_1 = J_1' \dot{J}_1 = 0 \tag{91}$$

$$C_2 = J_2' \dot{J}_2 = 0 \tag{92}$$

$$C = J_1 + J_2 = 0 \tag{93}$$

and

$$\tau_1 = J_1' f_1 = f_{11} \tag{94}$$

$$\tau_2 = J_2' f_2 = 0 \tag{95}$$

$$\tau = \tau_1 + \tau_2 = f_{11} \tag{96}$$

Thus the equation of motion in θ coordinates

$$M\ddot{\theta} + C\dot{\theta} = \tau \tag{97}$$

simplifies as follows

$$\ddot{\theta} = \frac{f_{11}}{m_1 + m_2} \tag{98}$$

It is interesting to study the effective forces $\hat{f}$ and constraint forces $\hat{e}$ underlying this solution:

$$\ddot{x}_1 = J_1 \ddot{\theta}_1 + \dot{J}_1 \dot{\theta}_1 = \begin{bmatrix} \frac{f_{11}}{m_1+m_2} \\ 0 \\ 0 \end{bmatrix} \tag{99}$$

Moreover

$$\hat{f}_1 = f_1 + e_1 = m_1 \ddot{x}_1 = \begin{bmatrix} \frac{m_1}{m_1+m_2} f_{11} \\ 0 \\ 0 \end{bmatrix} \tag{100}$$

Thus the constraint forces on the first particle are as follows

$$\hat{e}_1 = \hat{f}_1 - f_1 = \begin{bmatrix} -\frac{m_2}{m_1+m_2} f_{11} \\ -f_{12} \\ -f_{13} \end{bmatrix} \tag{101}$$

Using a similar argument we get

$$\hat{f}_2 = \begin{bmatrix} \frac{m_2}{m_1+m_2} f_{11} \\ 0 \\ 0 \end{bmatrix} \tag{102}$$

and

$$\hat{e}_2 = \hat{f}_2 - f_2 = \begin{bmatrix} \frac{m_1}{m_1+m_2} f_{11} \\ 0 \\ 0 \end{bmatrix} \tag{103}$$

2.3 Example: Multi Particle Pendulum

We model a solid pendulum as a collection of particles rigidly connected to each other via a common, massless rod (see Figure 5)

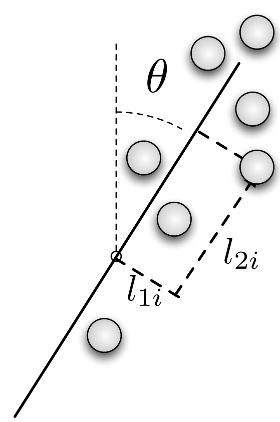

Figure 5: *Multi Particle Pendulum.*

$$x_i = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} l_{1i} \\ l_{2i} \\ 0 \end{bmatrix} = \begin{bmatrix} l_{1i}\cos(\theta) + l_{2i}\sin(\theta) \\ -l_{1i}\sin(\theta) + l_{2i}\cos(\theta) \\ 0 \end{bmatrix} \tag{104}$$

where $l_{1i}, l_{2,i}$ are the coordinates of a point mass with respect to the rotating pendulum. Thus

$$J_i = \begin{bmatrix} -l_{1i}\sin(\theta) + l_{2i}\cos(\theta) \\ -l_{1i}\cos(\theta) - l_{2i}\sin(\theta) \\ 0 \end{bmatrix} \tag{105}$$

$$\dot{J}_i = \dot{\theta} \begin{bmatrix} -l_{1i}\cos(\theta) - l_{2i}\sin(\theta) \\ l_{1i}\sin(\theta) - l_{2i}\cos(\theta) \\ 0 \end{bmatrix} \tag{106}$$

Therefore

$$M_i = m_i J_i' J_i = m_i \|l_i\|^2 \tag{107}$$

$$C_i = m_i J_i' \dot{J}_i = 0 \tag{108}$$

$$\tau_i = J_i' f_i = J_i' \begin{bmatrix} 0 \\ -m_i g \\ 0 \end{bmatrix} = m_i \big(l_{1i}\cos(\theta) + l_{2i}\sin(\theta)\big) g \tag{109}$$

Thus

$$M = \sum_i M_i = \sum m_i l_i^2 \tag{110}$$

$$C = \sum_i C_i = 0 \tag{111}$$

$$\tau = \sum_i \tau_i = \sum_i m_i \Big(l_{1i}\cos(\theta) + l_{2i}\sin(\theta)\Big) g \tag{112}$$

Applying Newton's law and solving for $\ddot{\theta}$

$$\ddot{\theta} = g \frac{\sum_i m_i \Big(l_{1i}\cos(\theta) + l_{2i}\sin(\theta)\Big)}{\sum m_i l_i^2} \tag{113}$$

For the simple pendulum case (single particular along the main axis) we get

$$\ddot{\theta} = \frac{g}{l}\sin(\theta) \tag{114}$$

2.4 Example: The Cart-Pole Problem (Inverted Pendulum)

We model the pole on a cart problem as a two particle system, the cart with mass m_1 and the pole, with mass m_2 (see Figure 6). The motion of the cart is constrained as follows

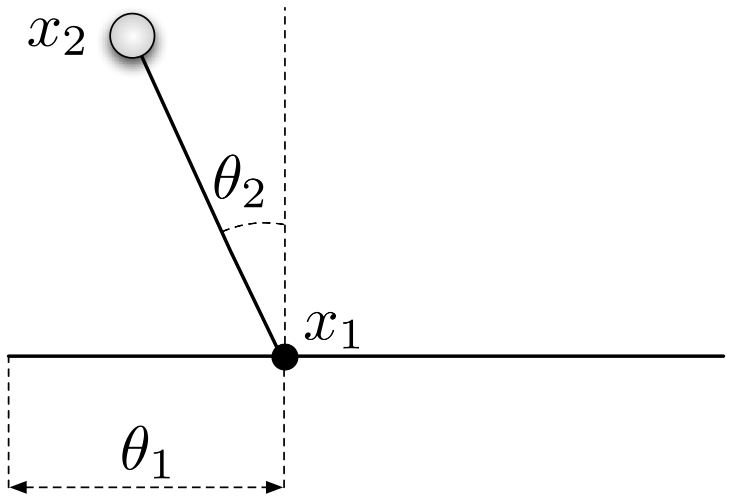

Figure 6: *Cart-Pole Problem.*

$$x_1 = \begin{bmatrix} \theta_1 \\ 0 \\ 0 \end{bmatrix} \tag{115}$$

and the motion of the pole is constrained as follows

$$x_2 = \begin{bmatrix} \theta_1 + l \sin\theta_2 \\ l\cos\theta_2 \\ 0 \end{bmatrix} \tag{116}$$

Where θ_1 is the location of the cart on the horizontal plane, and θ_2 the angle of the pole with respect to the vertical line. We first calculate the Jacobian and Jacobian velocity for particle 1

$$J_1 = \frac{\partial x_1}{\partial \theta'} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{117}$$

$$\dot{J}_1 = \sum_i \dot{\theta}_i \frac{\partial J_1}{\theta_i} = 0 \tag{118}$$

Thus

$$M_1 = m_1 J_1' J_1 = m_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{119}$$

$$C_1 = 0 \tag{120}$$

$$\tau_1 = J' \begin{bmatrix} f - \kappa_1 \dot{\theta}_1 \\ -m_1 g \\ 0 \end{bmatrix} = \begin{bmatrix} f - \kappa_1 \dot{\theta}_1 \\ 0 \end{bmatrix} \tag{121}$$

where f is the horizontal force applied to the cart and $\kappa_1 \dot{\theta}_1$ is a viscous friction force. Regarding the second particle (the pole)

$$J_2 = \frac{\partial x_2}{\partial \theta'} = \begin{bmatrix} 1 & l \cos\theta_2 \\ 0 & -l \sin\theta_2 \\ 0 & 0 \end{bmatrix} \tag{122}$$

$$\dot{J}_2 = \sum_{i=1}^{2} \dot{\theta}_i \frac{\partial J_2}{\partial \theta_i} = \dot{\theta}_2 \begin{bmatrix} 0 & -l \sin\theta_2 \\ 0 & -l \cos\theta_2 \\ 0 & 0 \end{bmatrix} \tag{123}$$

Thus

$$M_2 = m_2 J_2' J_2 = m_2 \begin{bmatrix} 1 & l \cos\theta_2 \\ l \cos\theta_2 & l^2 \end{bmatrix} \tag{124}$$

$$C_2 = m_2 J_2' \dot{J}_2 = -m_2 \begin{bmatrix} 0 & l \dot{\theta}_2 \sin\theta_2 \\ 0 & l^2 \end{bmatrix} \tag{125}$$

$$\tau_2 = J_2' \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\kappa_2 \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ m_2 g \sin\theta_2 - \kappa_2 \dot{\theta}_2 \end{bmatrix} \tag{126}$$

where $\kappa_2 \dot{\theta}_2$ is the rotational friction. Note we don't need to pre-multiply times J_2' because the frictional force operates directly on θ_2 , not on x_2. Thus, the equations of motion look as follows

$$M\ddot{\theta} + C\dot{\theta} = \tau \tag{127}$$

where

$$M = M_1 + M_2 = \begin{bmatrix} m_1 + m_2 & m_2 l \cos\theta_2 \\ m_2 l \cos\theta_2 & m_2 l^2 \end{bmatrix} \tag{128}$$

$$C = C_1 + C_2 = \begin{bmatrix} 0 & -l m_2 \dot\theta_2 \sin\theta_2 \\ 0 & -m_2 \dot\theta_2 l^2 \end{bmatrix} \tag{129}$$

$$\tau = \tau_1 + \tau_2 = \begin{bmatrix} f - \kappa_1 \dot\theta_1 \\ m_2 g \sin\theta_2 - \kappa_2 \dot\theta_2 \end{bmatrix} \tag{130}$$

Expanding (127) we get the standard cart-wheel equations

$$(m_1 + m_2)\ddot\theta_1 + m_2 l \cos\theta_2 \ddot\theta_2 - l m \dot\theta_2^2 \sin\theta_2 = f - \kappa_1 \dot\theta_1 \tag{131}$$

$$(m_2 l \cos\theta_2)\ddot\theta_1 + m_2 l^2 \ddot\theta_2 - m_2 \theta_2^2 l^2 = m_2 g \sin\theta_2 - \kappa_2 \dot\theta_2 \tag{132}$$

2.5 Example: Reaction Wheel Pendulum

Consider a rotating wheel of radius r attached to a rotating pole of length l. The angle of rotation of the arm with respect to the vertical is θ_1. The angle of rotation of the wheel with respect to the arm is θ_2. Note we define the angle of the wheel relative to the arm. This complicates the derivations a bit but it is worth it because motor encoders encode the angle of rotation with respect to the arm. In addition frictional forces in the joint between the arm and the wheel are due to changes in the relative angle between the wheel and the arm.

We model the wheel as a collection of particles separated from the center by an equal radius r and each of them with a different angular offset α_i with respect to a reference point in the wheel (see Figure 7). We model the arm as a collection of particles of mass m_i located along the line from the origin to the center of the wheel, and with distance l_i from the origin.

Particles in the Wheel The location of each particle in the wheel satisfies the following constraint

$$x_i = l \begin{bmatrix} \sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix} + r \begin{bmatrix} \sin(\theta_1 + \theta_2 + \alpha_i) \\ \cos(\theta_1 + \theta_2 + \alpha_i) \\ 0 \end{bmatrix} \tag{133}$$

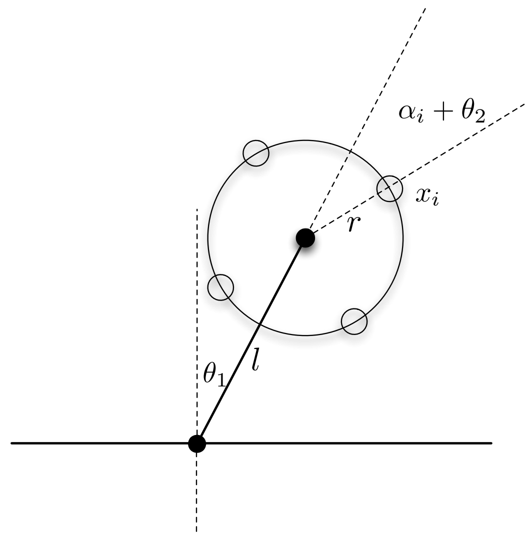

Figure 7: *Reaction Wheel.*

First we will compute the inertial matrix M_w for the wheel. The Jacobian for particles in the the wheel is as follows

$$J_i = \frac{\partial x_i}{\partial \theta'} = \begin{bmatrix} l\cos\theta_1 + r\cos(\theta_1+\theta_2+\alpha_i), & r\cos(\theta_1+\theta_2+\alpha_i) \\ -l\sin\theta_1 - r\sin(\theta_1+\theta_2+\alpha_i), & -r\sin(\theta_1+\theta_2+\alpha_i) \\ 0 & 0 \end{bmatrix} \tag{134}$$

Thus

$$\begin{aligned}(J_i'J_i)_{1,1} &= l^2 + r^2 + 2lr\Big(\cos(\theta_1)\cos(\theta_1+\theta_2+\alpha_i) + \sin(\theta_1)\sin(\theta_1+\theta_2+\alpha_i)\Big) \\ &= l^2 + r^2 + 2lr\cos(\alpha_i+\theta_2) \end{aligned} \tag{135}$$

where we used the fact that

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b) \tag{136}$$

$$(J_i'J_i)_{2,2} = r^2 \tag{137}$$

$$\begin{aligned}(J_i'J_i)_{1,2} &= r^2 + lr\Big(\cos(\theta_1)\cos(\theta_1+\theta_2+\alpha_i) + \sin(\theta_1)\sin(\theta_1+\theta_2+\alpha_i)\Big) \\ &= r^2 + lr\cos(\alpha_i+\theta_2) \end{aligned} \tag{138}$$

Thus the moment of inertial matrix for wheel particles looks as follows

$$M_i = m_i J_i' J_i = m_i \begin{bmatrix} l^2 + r^2 + 2lr\cos(\alpha_i + \theta_2) & r^2 + lr\cos(\alpha_i + \theta_2) \\ r^2 + lr\cos(\alpha_i + \theta_2) & r^2 \end{bmatrix} \tag{139}$$

Adding up the inertial matrices of the particles, we get the overall inertial mattrix of the wheel

$$M_w = \sum_i M_i = m_w \begin{bmatrix} l^2 + r^2 & r^2 \\ r^2 & r^2 \end{bmatrix} \tag{140}$$

where

$$m_w \stackrel{\text{def}}{=} \sum_{i \in \text{wheel}} m_i \tag{141}$$

In the last step we assumed that for each particle i there is another particle j such that $m_j \cos(\alpha_j + \theta_2) = -m_i \cos(\alpha_i + \theta_2)$.

Next we compute the Coriolis matrix for the wheel particles

$$\dot{J}_i = \dot{\theta}_1 \frac{\partial J_i}{\partial \theta_1} + \dot{\theta}_2 \frac{\partial J_i}{\partial \theta_2} = \tag{142}$$

where

$$\frac{\partial J_i}{\partial \theta_1} = \begin{bmatrix} -l\sin\theta_1 - r\sin(\theta_1 + \theta_2 + \alpha_i) & -r\sin(\theta_1 + \theta_2 + \alpha_i) \\ -l\cos\theta_1 - r\cos(\theta_1 + \theta_2 + \alpha_i) & -r\cos(\theta_1 + \theta_2 + \alpha_i) \\ 0 & 0 \end{bmatrix} \tag{143}$$

$$\frac{\partial J_i}{\partial \theta_2} = \begin{bmatrix} -r\sin(\theta_1 + \theta_2 + \alpha_i) & -r\sin(\theta_1 + \theta_2 + \alpha_i) \\ -r\cos(\theta_1 + \theta_2 + \alpha_i) & -r\cos(\theta_1 + \theta_2 + \alpha_i) \\ 0 & 0 \end{bmatrix} \tag{144}$$

After some algebra, it can be shown that

$$J_i' \frac{\partial J_i}{\partial \theta_1} = \begin{bmatrix} 0 & lr\sin(\alpha_i + \theta_2) \\ -lr\sin(\alpha_i + \theta_2) & 0 \end{bmatrix} \tag{145}$$

$$J_i' \frac{\partial J_i}{\partial \theta_2} = \begin{bmatrix} lr\sin(\alpha_i + \theta_2) & lr\sin(\alpha_i + \theta_2) \\ 0 & 0 \end{bmatrix} \tag{146}$$

Thus

$$C_i = m_i \begin{bmatrix} \dot{\theta}_2 lr\sin(\alpha_i + \theta_2) & (\dot{\theta}_1 + \dot{\theta}_2) lr\sin(\alpha_i + \theta_2) \\ -\dot{\theta}_1 lr\sin(\alpha_i + \theta_2) & 0 \end{bmatrix} \tag{147}$$

The Coriolis matrix for the entire wheel follows

$$C_w = \sum_i C_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{148}$$

where we used the fact that for each particle i in the wheel there is a particle j such that $m_i \sin(\alpha_i + \theta_2) = -m_j \sin(\alpha_j + \theta_2)$.

Particles in the Arm: In this case

$$x_i = l_i \begin{bmatrix} \sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix} \tag{149}$$

where l_i is the distance from the origin to the location of the particle in the main arm. Thus the Jacobian matrix for arm particles looks as follows

$$J_i = \frac{\partial x_i}{\partial \theta'} = \begin{bmatrix} l_i \cos\theta_1 & 0 \\ -l_i \sin\theta_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{150}$$

and the inertial matrix looks as follows

$$M_i = m_i J_i' J_i = m_i \begin{bmatrix} l_i^2 & 0 \\ 0 & 0 \end{bmatrix} \tag{151}$$

Adding up the inertial matrices of the arm particles we obtain the overall inertial matrix for the arm

$$M_a = \sum_i M_i = \begin{bmatrix} \bar{m}_a & 0 \\ 0 & 0 \end{bmatrix} \tag{152}$$

where

$$\bar{m}_a = \sum_{i \in \text{arm}} m_i l_i^2 \tag{153}$$

To compute the Coriolis matrix we first get the velocity of the Jacobian

$$\dot{J}_i = \sum_{j=1}^{2} \dot{\theta}_j \frac{\partial J_i}{\partial \theta_j} = \dot{\theta}_1 \begin{bmatrix} -l_i \sin\theta_1 & 0 \\ -l_i \cos\theta_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{154}$$

Thus

$$C_i = m_i J_i' \dot{J}_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{155}$$

and

$$C_a = \sum_i C_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{156}$$

Thus the overall inertial matrix is as follows

$$M = M_a + M_w = \begin{bmatrix} m_w\,(l^2 + r^2) + \bar{m}_a & m_w r^2 \\ m_w r^2 & m_w\, r^2 \end{bmatrix} \tag{157}$$

where

$$m_w \stackrel{\text{def}}{=} \sum_{i \in \text{wheel}} m_i \tag{158}$$

$$\bar{m}_a = \sum_{i \in \text{arm}} m_i l_i^2 \tag{159}$$

The overall Coriolis matrix is zero

$$C = C_a + C_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{160}$$

In addition to the torques produced by the motor we need to consider the effect of gravity on each particle.

Gravity: For particles in the wheel the torque due to gravity is as follows

$$\tau_i = J_i' \begin{bmatrix} 0 \\ -m_i g \\ 0 \end{bmatrix} = m_i g \begin{bmatrix} l \sin(\theta_1) + r \sin(\theta_1 + \theta_2 + \alpha_i) \\ r \sin(\theta_1 + \theta_2 + \alpha_i) \end{bmatrix} \tag{161}$$

and for particles in the arm

$$\tau_i = J_i' \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} = m_i l_i g \begin{bmatrix} \sin(\theta_1) \\ 0 \end{bmatrix} \tag{162}$$

Thus the total torque due to gravity is as follow

$$\tau_g = \sum_{i \in \text{wheel}} \tau_i + \sum_{i \in \text{arm}} \tau_i = \Big(\sum_{i \in \text{arm}} m_i l_i + l m_w \Big) g \begin{bmatrix} \sin\theta_1 \\ 0 \end{bmatrix} \tag{163}$$

where we used the fact that for each particle i in the wheel with there is another particle j such that $m_i \sin(\theta_i + \theta_2 + \alpha_i) = -m_j \cos(\theta_1 + \theta_2 + \alpha_j)$.

DC Motors: Typically the wheel is powered by a DC motor whose input voltage is controlled using standard electronics. The equations of motion for DC motors are as follows

$$V = L\frac{dI}{dt} + RI + k_b\dot{\theta}_2 \tag{164}$$

$$\tau = k_T I \tag{165}$$

where V is the voltage applied to the motor, L is the motor inductance, I the current through the motor windings, R the motor winding resistance, k_b the motor's back electro magnetic force constant, $\dot{\theta}_2$ the rotor's angular velocity, and k_T the motor's torque constant. The electrical time constant for the current is typically in the order of a few hundreds of a millisecond, and the mechanical time constant is in the order of tens of a millisecond. In practice we can approximate these equations by treating the electrical time constant L/R as if it were zero. Under this approximation

$$V = RI + k_b\dot{\theta}_2 \tag{166}$$

$$\tau = \frac{k_T V}{R} - \frac{k_b k_T}{R}\dot{\theta}_2 \tag{167}$$

The torques due to viscous friction take the following form

$$\tau_v = -\begin{bmatrix} \nu_1 & \\ 0 & \nu_2 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{168}$$

where ν_1, ν_2 are viscosity coefficients. Bringing it all together we get the equations of motion for the reaction wheel pendulum

$$\begin{bmatrix} 0 \\ k_T/R \end{bmatrix} V - \begin{bmatrix} \nu_1, & 0 \\ 0, & \nu_2 + \frac{k_b k_T}{R} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \tau_g = \begin{bmatrix} m_w\,(l^2 + r^2) + \bar{m}_a & m_w r^2 \\ m_w r^2 & m_w\, r^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$m_w = \textstyle\sum_{i\in\text{wheel}} m_i$$

$$\bar{m}_a = \textstyle\sum_{i\in\text{arm}} m_i l_i^2$$

$$\tau_g = g\left(\textstyle\sum_{i\in\text{arm}} m_i l_i + m_w\, l\right)\begin{bmatrix} \sin\theta_1 \\ 0 \end{bmatrix} \tag{169}$$

Parameters of Physical Model:

- m_w. Mass of wheel.
- r. Radius of wheel
- l. Length of pendulum
- g. Gravitational acceleration constant.
- $\bar{m}_a = \sum_{i \in arm} m_i l_i^2$
- $\sum_{i \in arm} m_i l_i$
- k_T Motor torque Constant
- k_b Motor back emf constant. When using SI Units $k_b = k_T$.
- R Motor electrical resistance.
- ν_1 Viscous friction of pendulum link.
- ν_2 Viscous friction of wheel link.

Linear Regression Model Note according to the model, the acceleration $\ddot{\theta}_1$ and $\ddot{\theta}_2$ are linear combinations of $[V, \dot{\theta}_1, \dot{\theta}_2, \sin(\theta_1)]$ where V is the voltage applied to the motor.

2.6 Dynamic Constraints

Up to now we have assumed that the constraint h does not change. Here we study a generalization in which h may be dynamic. To do so we let

$$x_t = h(u_t, \theta_t) \tag{170}$$

where u is a known function of time. For example, if we let $u_t = t$ then h is a function of time. In this case

$$\dot{x} = J_\theta \dot{\theta} + J_u \dot{u} \tag{171}$$

$$\ddot{x} = \dot{J}_\theta \dot{\theta} + J_\theta \ddot{\theta} + \dot{J}_u \dot{u} + J_u \ddot{u} \tag{172}$$

Applying Newton's law

$$f + e = m\ddot{x} \tag{173}$$

$$f + e = m\Big(\dot{J}_\theta \dot{\theta} + J_\theta \ddot{\theta} + \dot{J}_u \dot{u} + J_u \ddot{u}\Big) \tag{174}$$

or equivalently

$$f - mJ_\theta \dot{\theta} - m\dot{J}_u \dot{u} - mJ_u \ddot{u} + e = mJ_\theta \ddot{\theta} \tag{175}$$

Thus the problem is identical to the previous case but with additional apparent forces.

2.6.1 Example: Pendulum with Moving Base

Consider a point with mass m with the following constrained motion

$$x = h(u, \theta) = l \begin{bmatrix} \sin\theta \\ \cos\theta \\ 0 \end{bmatrix} + u \tag{176}$$

Where l is the fixed length of the pendulum and u is a predetermined function of time. The equation implements a point mass pendulum with a rotation point u that moves as a function of time in a predetermined manner (see Figure 8). Note the trajectory of the rotation point u, the base of the pendulum, is predetermined and cannot be changed by external forces. On the other hand the pendulum's angle of rotation can be changed by external forces f applied to the pendulum's point mass. Note

$$J_\theta = \frac{\partial x}{\partial \theta'} = l \begin{bmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{bmatrix} \tag{177}$$

$$M_\theta = mJ'_\theta J_\theta = ml^2 \tag{178}$$

$$\dot{J}_\theta = \dot{\theta}\frac{\partial J}{\partial \theta} = \dot{\theta} \begin{bmatrix} -\sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \tag{179}$$

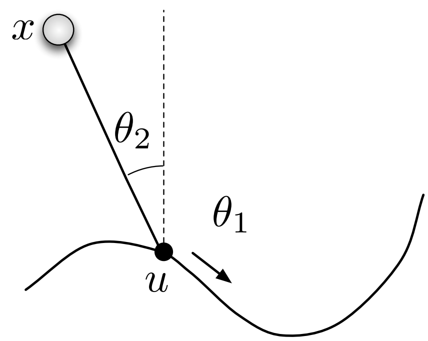

Figure 8: *Pendulum with Moving Base.*

Thus

$$C_\theta = mJ'\dot{J} = 0 \tag{180}$$

Moreover,

$$J_u = \mathbf{1} \tag{181}$$
$$\dot{J}_u = \mathbf{0} \tag{182}$$

Thus

$$M_u = mJ'_\theta J_u = mJ'_\theta \mathbf{1} \tag{183}$$
$$C_u = \mathbf{0} \tag{184}$$

Regarding the external forces, first we have the generalized force due to gravity

$$\tau_1 = J'_\theta \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} = mg \sin\theta \tag{185}$$

In addition we can have rotational friction $-\kappa\dot{\theta}$, proportional to the rotational velocity. Thus Gauss' equations of motion take the following form

$$ml^2\ddot{\theta} = (mg\sin\theta) - \kappa\dot{\theta} - \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \ddot{u} \tag{186}$$

Rearranging terms

$$\ddot{\theta} = \frac{g}{l}\sin(\theta) + \frac{\ddot{u}_2 \sin\theta}{l} - \frac{\ddot{u}_1 \cos\theta}{l} - \frac{\kappa}{ml^2}\dot{\theta} \tag{187}$$

Note if the base is fixed, i.e., u is not a function of time, and there is no rotational friction, we recover the simple pendulum equation

$$\ddot{\theta} = \frac{g}{l}\sin\theta \tag{188}$$

3 The Lagrangian Approach

Let L represent the difference between the kinetic energy of a particle and the total work produced by the forces applied to that particle

$$L = K - W \tag{189}$$

$$K = \frac{1}{2}m\, \dot{x}'\dot{x} \tag{190}$$

$$W = \int_0^t f_s' dx_s \tag{191}$$

Note

$$\frac{\partial W}{\partial x_i} = \frac{\partial}{\partial x_i} \sum_k \int_0^t f_k(s) dx_k(s) = f_k(t) \tag{192}$$

Thus

$$\frac{\partial W}{\partial x} = f \tag{193}$$

Moreover

$$\frac{\partial W}{\partial \dot{x}} = 0 \tag{194}$$

$$\frac{\partial K}{\partial x} = 0 \tag{195}$$

$$\frac{\partial K}{\partial \dot{x}} = m\dot{x} \tag{196}$$

Thus Newton's equation of motion

$$\frac{d}{dt} m\dot{x} = f \tag{197}$$

can also be expressed as follows

$$\frac{d}{dt}\frac{\partial L(x,\dot{x})}{\partial \dot{x}} - \frac{\partial L(x,\dot{x})}{\partial x} = 0 \tag{198}$$

Note as standard Euler-Lagrange equation from calculus of variations. It shows that trajectories that obey Newton's law minimize the following action integral

$$I = \int_0^t L(x_s, \dot{x}_s)\; ds \tag{199}$$

3.0.2 The constrained case

We now consider the cases in which x is a function of θ. In this case we have that

$$x = h(\theta) \tag{200}$$

$$\dot{x} = J\dot{\theta} \tag{201}$$

$$K = \frac{1}{2} m\dot{\theta}' J' J \dot{\theta} = \frac{1}{2}\dot{\theta}' M \dot{\theta} \tag{202}$$

$$W = \int_0^t f_s' J_s d\theta_s \tag{203}$$

As we will show soon it turns out that the Lagrangian equations of motion also apply to the constrained case, i.e.,

$$\boxed{\frac{d}{dt}\frac{\partial L(\theta,\dot{\theta})}{\partial \dot{\theta}} - \frac{\partial L(\theta,\dot{\theta})}{\partial \theta} = 0} \tag{204}$$

showing that the trajectory of θ minimizes the action integral

$$I = \int_0^t L(\theta_s, \dot{\theta}_s)\; ds \tag{205}$$

Note

$$\frac{\partial K}{\partial \theta_i} = \frac{1}{2}m\frac{\partial u'u}{\partial \theta_i} = \frac{1}{2}m\frac{\partial u'}{\partial \theta_i}\frac{\partial u'u}{\partial u} = m\frac{\partial \dot\theta' J'}{\partial \theta_i}J\dot\theta' = m\dot\theta'\frac{\partial J'}{\partial \theta_i}J\dot\theta \tag{206}$$

where $u = J\dot\theta$. Note we are taking the derivative with respect to θ holding $\dot\theta$ constant. Moreover

$$\frac{\partial W}{\partial \theta_i} = \frac{\partial}{\partial \theta_i}\sum_k \int_0^t (f'J)_k \, d\theta_k = (f'J)_i \tag{207}$$

Thus

$$\frac{\partial W}{\partial \theta} = J'f \tag{208}$$

Moreover

$$\frac{\partial W}{\partial \dot\theta} = 0 \tag{209}$$

$$\frac{\partial K}{\partial \dot\theta} = M\dot\theta \tag{210}$$

$$\tag{211}$$

The derivative of the kinetic energy K with respect to $\dot\theta$ is known as the **momentum** (aka generalized momentum) with respect to θ

$$l = \frac{\partial K}{\partial \dot\theta} = M\dot\theta \tag{212}$$

$$\tag{213}$$

The derivative of the work W with respect to θ is known as the **generalized force** τ (or torque) on θ

$$\tau = \frac{\partial}{\partial \theta} = J'f \tag{214}$$

If the force f is caused by a potential field V then

$$J'f = \frac{\partial x'}{\partial \theta}\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \theta} \tag{215}$$

Taking derivatives with respect to time

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}} = \frac{dl}{dt} = M\ddot{\theta} + \frac{dM}{dt}\dot{\theta} = M\ddot{\theta} + J'\dot{J}\dot{\theta} + \dot{J}'J\dot{\theta} \tag{216}$$

Thus the Lagrangian equation of motion looks as follows

$$\frac{dl_i}{dt} = \tau_i + m\dot{\theta}'\,\frac{\partial J'}{\partial \theta_i}J\dot{\theta} \tag{217}$$

for all i. Equivalently

$$\left(M\ddot{\theta} + J'\dot{J}\dot{\theta} + \dot{J}'J\dot{\theta} - J'f\right)_i - m\dot{\theta}'\,\frac{\partial J'}{\partial \theta_i}J\dot{\theta} = 0 \tag{218}$$

for all i. We will just show that this equation is identical Gauss's equation of motion.

Proof: The Newtonian equations of motion are

$$M\ddot{\theta} + J'\dot{J}\dot{\theta} = J'f \tag{219}$$

Thus, we just need to show that for all k

$$\left(\dot{J}'J\dot{\theta}\right)_k = \dot{\theta}'\,\frac{\partial J'}{\partial \theta_k}J\dot{\theta} \tag{220}$$

First note

$$\left(\frac{\partial J'}{\partial \theta_k}J\right)_{ij} = \sum_l \frac{\partial J_{li}}{\partial \theta_k}J_{lj} \tag{221}$$

Thus

$$\dot{\theta}'\,\frac{\partial J'}{\partial \theta_i}J\dot{\theta} = \sum_i\sum_j\sum_l \dot{\theta}_i\dot{\theta}_j\frac{\partial J_{li}}{\partial \theta_k}J_{lj} \tag{222}$$

$$= \sum_i\sum_j\sum_l \dot{\theta}_i\dot{\theta}_j\frac{\partial^2 x_l}{\partial \theta_i\partial \theta_k}\frac{\partial x_l}{\partial \theta_j} \tag{223}$$

Now note

$$\left(\dot{J}'J\dot{\theta}\right)_k = \sum_j \left(\dot{J}J\right)_{kj}\dot{\theta}_j \tag{224}$$

where

$$\left(\dot{J}J\right)_{kj} = \sum_l \dot{J}_{lk} J_{lj} = \sum_l \sum_i \dot{\theta}_i \frac{\partial J_{lk}}{\partial \theta_i} J_{lj} = \sum_l \sum_i \dot{\theta}_i \frac{\partial^2 x_l}{\partial \theta_i \partial \theta_k} \frac{\partial x_l}{\partial \theta_j} \tag{225}$$

Thus

$$\left(\dot{J}' J \dot{\theta}\right)_k = \sum_j \sum_l \sum_i \dot{\theta}_i \, \frac{\partial^2 x_l}{\partial \theta_i \partial \theta_k} \frac{\partial x_l}{\partial \theta_j} \, \dot{\theta}_j \tag{226}$$

3.0.3 Alternative Form of Lagrangian Equation

The following form of the Lagrangian equation is sometimes popular:

$$M\ddot{\theta} + C\dot{\theta} = \tau \tag{227}$$

where

$$C_{ij} = \sum_k h_{ijk} \dot{\theta}_j \dot{\theta}_k \tag{228}$$

$$h_{ijk} = \frac{\partial M_{ij}}{\partial \theta_k} - \frac{1}{2} \frac{\partial M_{jk}}{\partial \theta_i} \tag{229}$$

It can be shown that $C = J'\dot{J}$ thus reducing to the standard Newtonian form. The terms $h_{ijj}\dot{\theta}_j^2$ represent the centrifugal force induced on θ_i by $\dot{\theta}_j$. The terms $h_{ijk}\dot{\theta}_j\dot{\theta}_k$ represent the Coriolis force on θ_i induced by the velocities $\dot{\theta}_j$, $\dot{\theta}_k$. Thus, the term C_{ij} represents the sum of Coriolis and centrifugal forces on θ_i induced by the the velocity θ_j.

4 Rotational Motion: Single Particle

Rotations are one of the most important forms of constrained motion. We will represent of rotations as matrix exponentials.We recommend the reader to refer to the primer on rotations available at mplab.ucsd.edu. Below are the most important properties of matrix exponential.

4.1 Matrix Exponentials and Rotations

A rotation matrix r can be represented as follows

$$r \stackrel{\text{def}}{=} e^{R[\theta]} \tag{230}$$

where the function $R : \mathcal{R}^3 \to \mathcal{R}^3 \otimes \mathcal{R}^3$ is defined as follows

$$R[\theta] \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \tag{231}$$

It can be shown that

$$e^{R[\theta]} = \mathbf{1} + R[\frac{\theta}{|\theta|}]\sin(|\theta|) + R^2[\frac{\theta}{|\theta|}](1 - \cos(|\theta|)) \tag{232}$$

where the function R^2 is defined as follows

$$R^2[x] \stackrel{\text{def}}{=} R[x]R[x] = xx' - \mathbf{1}(x'x) \tag{233}$$

$$= \begin{bmatrix} -x_3^2 - x_2^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & -x_2^2 - x_3^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & -x_1^2 - x_2^2 \end{bmatrix} \tag{234}$$

(235)

Note in the matrix exponential representation, the direction of the θ vector is the axis around which the rotation happens, and the magnitude $|\theta|$ of θ is the amount of rotation about that axis. The sign for the rotation follows the right hand rule for rotations. We orient the right thumb with the axis of rotation. A positive rotation follows the curl of the other four fingers of the right hand. Since we can change the direction and magnitude of θ we can think of x as the endpoint of an arm of length b attached to a ball socket fixed at the origin.

4.1.1 Rotations About the Main Axis

The matrix for rotating θ degrees about the horizontal (x), vertical (y) and depth (z) axis are as follows

$$r_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{236}$$

$$r_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{237}$$

$$r_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{238}$$

Example: Let

$$u = \theta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{239}$$

In this case

$$r = e^{R[\theta u]} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{240}$$

Note the point $p = [1, 0, 0]$ along the horizontal axis, would rotate into the point $rp = [\cos\theta, \sin\theta, 0]'$, which is an θ degrees rotation of p around the depth axis in a counterclockwise direction.

4.2 Rotation Dynamics

Consider a particle of mass m whose location satisfies the following function

$$x = rb \tag{241}$$

where $b \in \mathcal{R}^3$ are constants a constant and r is a rotation matrix, parameterized as a matrix exponential of $\theta \in \mathcal{R}^3$

Using the properties of matrix exponentials (see primer on rotations) we get

$$J = \frac{\partial x}{\partial \theta'} = R'[x] \tag{242}$$

$$\dot{J} = \frac{\partial J}{\partial t} = R'[\dot{x}] \tag{243}$$

Thus the moment of inertia matrix is as follows

$$M = mJ'J = mR[x]R'[x] = -mR^2[x] \tag{244}$$

Note

$$-R^2[x] = b'r'rb\mathbf{1} - rbb'r' = b'b\mathbf{1} - rbb'r' = r\bar{M}r' \tag{245}$$

where

$$\bar{M} = -R^2[b] \tag{246}$$

and **1** is the 3×3 identity matrix.

The Coriolis matrix is as follows

$$C = mJ'\dot{J} = mR[x]R'[\dot{x}] \tag{247}$$

where

$$\dot{x} = J\dot{\theta} = R'[x]\dot{\theta} \tag{248}$$

The momentum with respect to θ is known as the **angular momentum**.

$$l = M\dot{\theta} \tag{249}$$

The force with respect to θ is known as the **torque**, defined as follows

$$\tau = J'f = R[x]f = x \times f \tag{250}$$

We will show now that in this case Newton's law

$$\tau = M\ddot{\theta} + C\dot{\theta} \tag{251}$$

can be expressed as follows

$$\boxed{\tau = \frac{d}{dt}l} \tag{252}$$

or as follows

$$\boxed{\tau = M\ddot{\theta} + \dot{\theta} \times l} \tag{253}$$

Proof: Wit respect to (252) note

$$\frac{d}{dt}l = \frac{M\theta}{dt} = M\ddot{\theta} + \frac{dM}{dt}\dot{\theta} = M\ddot{\theta} + J'\dot{J}\dot{\theta} + \dot{J}'J\dot{\theta} \tag{254}$$

$$= M\ddot{\theta} + C\dot{\theta} + \dot{J}'J\dot{\theta} \tag{255}$$

Moreover

$$\dot{J}'J\dot{\theta} = R[\dot{x}]R'[x] = R[R'[x]\dot{\theta}]R'[x]\dot{\theta} = (\dot{\theta} \times x) \times (\dot{\theta} \times x) = 0 \tag{256}$$

With respect to (253) note

$$\dot{\theta} \times l = -l \times \dot{\theta} = R'[l]\ \dot{\theta} \tag{257}$$

and therefore we just need to show that

$$C = mR[x]R'[\dot{x}] = R'[l] \tag{258}$$

Note

$$C\dot{\theta} = mR[x]R'[\dot{x}]\dot{\theta} = -mR[x](\dot{x} \times \dot{\theta}) \tag{259}$$

$$= -mR[x]\Big((R'[x]\dot{\theta}) \times \dot{\theta}\Big) = mx \times \Big((x \times \dot{\theta}) \times \dot{\theta}\Big) \tag{260}$$

Moreover

$$\dot{\theta} \times L = -m\dot{\theta} \times (R[x]R[x]\dot{\theta}) = -m\dot{\theta} \times \Big(x \times (x \times \dot{\theta})\Big) \tag{261}$$

$$= mx \times \Big((x \times \dot{\theta}) \times \dot{\theta}\Big) = C\dot{\theta} \tag{262}$$

For the last step we used the following property of the cross product operator: For any vectors $a, b, c \in \mathcal{R}^3$

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0 \tag{263}$$

In our case we used $a = x,\ b = x \times \dot{\theta},\ c = \dot{\theta}$

4.3 Rotations about a fixed axis

In this case θ is a scalar representing the degree of rotation about a fixed axis u (see Figure 9)

$$x = rb = e^{R[\theta u]}b \tag{264}$$

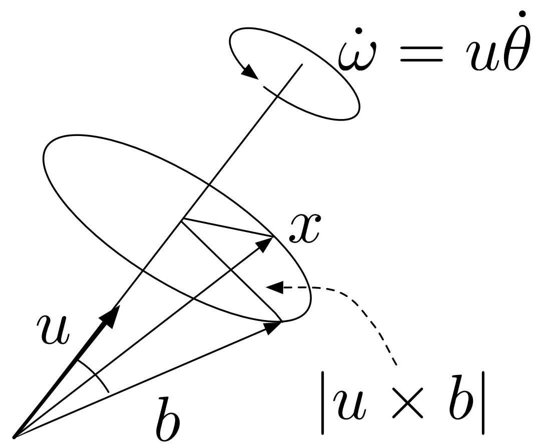

Figure 9: *Rotation about a fixed axis.*

$$J = \frac{\partial x}{\partial \theta u'} \frac{\partial \theta u}{\partial \theta} = R'[x]u \tag{265}$$

Thus

$$M = mJ'J = -mu'R^2[x]u = -mu'rR^2[b]r'u = u'\bar{M}u \tag{266}$$

where

$$\bar{M} = -R^2[b] \tag{267}$$

and we used the fact that $ru = r'u = u$ because u is the axis of rotation. Note

$$\begin{aligned} -u'R^2[b]u = u'\Big(b'b\mathbf{1} - bb'\Big)u = l^2 - u'bb'u = \|b\|^2 - (\|b\|\cos\theta)^2 \\ = (\|b\|\sin\theta)^2 = \|b \times u\|^2 \end{aligned} \tag{268}$$

where we used the fact that b is orthogonal to u. Thus in this case the moment of inertial looks as follows

$$M = u'\bar{M}u = m\|b \times u\|^2 \tag{269}$$

Regarding the Coriolis/Centrifugal/Centripetal term, note

$$\dot{J} = R'[\dot{x}]u \tag{270}$$

where

$$\dot{x} = J\dot{\theta} \tag{271}$$

Thus

$$\dot{J} = \dot{\theta}R'[R'[x]u]u = R[R[x]u]u = \dot{\theta}(x \times u) \times u \tag{272}$$

Thus

$$C = mJ'\dot{J} = m\dot{\theta}(x \times u)'\Big((x \times u) \times u\Big) = 0 \tag{273}$$

where we used the fact that $(x \times u) \times u$ is orthogonal to $x \times u$. Suppose a force f is applied on the rotating mass. The torque looks as follows

$$\tau = J'f = u'R[x]f = u'(x \times f) \tag{274}$$

Thus Newton's law in this case looks as follows

$$\boxed{u'\ (x \times f) = m\|b \times u\|^2\ \ddot{\theta}} \tag{275}$$

This models a 1 df joint. Explore how to best model a 2 and 3 df joint. The only way to do this is to mutiply 1 df joints, which amounts to the Euler Angle approach.

4.4 Apparent Forces Caused by Rotational Constraints

Consider a cannon attached to a rotating platform. The cannon fires a ball towards the center of the platform. We want to describe the trajectory of the cannon ball from the point of view of a reference frame that rotates with the platform. Note this is a non-inertial frame of reference. Let

$$x = h(u, \theta) = r\theta \tag{276}$$

where $r = e^{R[u]}$, and u follows a predetermined function of time that determines the 3D rotation of the reference frame. x has the space coordinates of

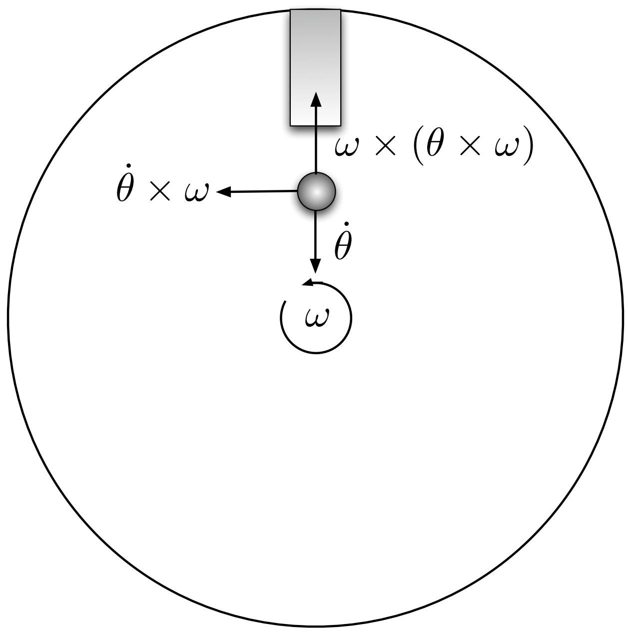

Figure 10: *A cannon attached to a rotating platform fires a ball towards the center of the platform. The ball exits the cannon with velocity $\dot\theta$ in rotating frame coordinates. From the rotating frame point of view the ball will appear to slow down due to the centrifugal force and deflect to the right (in platform centered coordinates) due to the Coriolis force.*

a point mass (e.g., the cannon ball), and θ the coordinates from the point of view of a rotating reference frame. The columns of r are a basis of the rotating frame of reference. Note

$$\dot x = R'[x]\dot u + r\dot\theta = R'[r\theta]\dot u + r\dot\theta \tag{277}$$

and therefore

$$\ddot x = R'[\dot x]\dot u + R'[x]\ddot u + R'[r\dot\theta]\dot u + r\ddot\theta \tag{278}$$

Using (277)

$$\ddot x = R'[R'[r\theta]\dot u]\dot u + R'[r\dot\theta]\dot u + R'[r\theta]\ddot u + R'[r\dot\theta]\dot u + r\ddot\theta \tag{279}$$

$$= -\dot u \times (r\theta \times \dot u) - 2(r\dot\theta) \times \dot u - r\theta \times \ddot u + r\ddot\theta \tag{280}$$

Newton's law of motion for general coordinate transformations looks as follows

$$J_\theta' f = m J_\theta' \ddot{x} \tag{281}$$

where

$$J_\theta = \frac{\partial x}{\partial \theta'} = r \tag{282}$$

and f is the force applied to the point mass, in space coordinates. Rearranging terms

$$\ddot{\theta} = r'\Big(\frac{f}{m} + \dot{u} \times ((r\theta) \times \dot{u}) + 2(r\dot{\theta}) \times \dot{u} + (r\theta) \times \ddot{u}\Big) \tag{283}$$

or equivalently

$$\ddot{\theta} = \Big(\frac{r' f}{m} + \omega \times (\theta \times \omega) + 2\dot{\theta} \times \omega + \theta \times \dot{\omega}\Big) \tag{284}$$

where

$$\omega = r'\dot{u} \tag{285}$$

$$\dot{\omega} = r'\ddot{u} \tag{286}$$

are the angular velocity and angular acceleration in the coordinates of the rotating frame of reference. Note the price of using the rotating frame of reference is that we get 3 new fictitious forces:

- **The centrifugal force:** $m\omega \times (\theta \times \omega)$
- **The Coriolis force**: $2m\dot{\theta} \times \omega$
- **The Euler force**: $\theta \times \dot{\omega}$

These forces are called "fictitious" because they are due to a kinematic effect, i.e., the change of a reference frame. If we insist on applying Newton's law to a non-inertial reference frame, then we need to add these extra terms to make sense of the observed motions. Note: I always have a problem with the word fictitious, since one can feel the centrifugal force in one's bones.

4.4.1 Example: Coriolis Fountain

This is a popular exhibition in many science museums. A water pipe attached to a rotating disk points towards the center of the disk. The situation is identical to the rotating cannon in Figure 4.4. As water flows out of the pipe the water is moving with velocity $\dot{\theta}$ pointing towards the center of the platform. This creates a Coriolis force tangential to the direction of rotation. In addition there is an outwards centrifugal force $\omega \times (\theta \times \omega)$. Because of the Coriolis force the water is deflected in the direction of the rotation and ends up in front of the pipe. This violates our intuitions that expect for the water to trail behind the rotating pipe, thus making for a great science exhibit. One way to think about it is that as the water gets closer to the center of the platform its angular velocity needs to increase so as to preserve angular momentum.

Note the initial velocity $\dot{\theta}$ of the ball's coordinates from the platform's reference frame points towards the center of the platform. This is because the cannon points towards the center of the platform. The direction vector of the cannon is constant in platform coordinates and points towards the center of the platform. However note that the initial velocity of the absolute space coordinates have a tangential component in the direction of rotation

$$\dot{x} = -(r\theta) \times \dot{u} + r\dot{\theta} \tag{287}$$

Note however if the water pipe was anchored at the center of the ring pointing towards the periphery, then the Coriolis force would point against the direction of rotation and the water would trail the pipe.

4.5 Motion in Polar Coordinates

Any motion can be expressed as a combination of rotational and tangential motion with respect to a stationary point. Consider a particle with mass m and location x. We can choose an arbitrary fixed reference point o and describe the motion of x in terms of a rotation and scaling with respect to the reference point (see Figure 11)

$$x - o = \alpha e^{R[\beta]}(b - o) \tag{288}$$

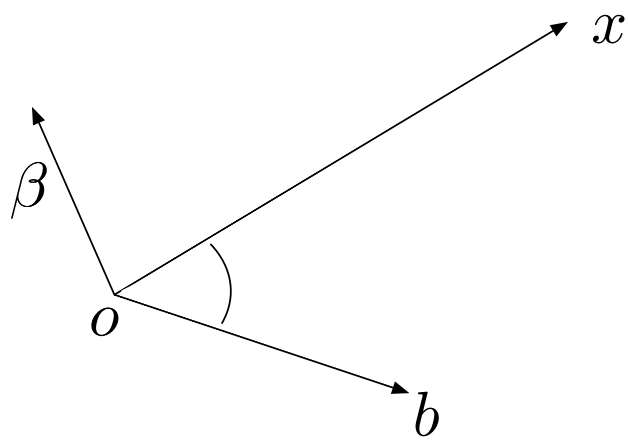

Figure 11: *Any motion can be described as a rotation and scale about a reference point o.*

Note

$$J_\alpha = \frac{\partial x}{\partial \alpha} = \frac{(x-o)}{\alpha} \tag{289}$$

$$J_\beta = \frac{\partial x}{\partial \beta} = R'[x-o] \tag{290}$$

Thus

$$\dot{x} = J_\alpha \dot{\alpha} + J_\beta \dot{\beta} = \frac{x-o}{\alpha}\dot{\alpha} + \dot{\beta} \times (x-o) \tag{291}$$

The velocity $\dot{x}$ can be decomposed into a **radial velocity**

$$\frac{x-o}{\alpha}\dot{\alpha} \tag{292}$$

that points in the direction of the particle, and a **tangential velocity**

$$\dot{\beta}(x-o) \tag{293}$$

orthogonal to the radial velocity.

5 Rotational Motion: Multiple Particles

Consider a collection of particles each with mass m_i and with positions satisfying the following equation

$$x_i = rb_i = e^{R[\theta]} b_i \tag{294}$$

where b_i is a constant vector and θ a time varying vector.

5.1 Inertial Matrix

From the past section we have that

$$M_i = -m_i R^2[x_i] = -m_i r R^2[b_i] r' \tag{295}$$

and

$$M = \sum_i M_i = r\bar{M}r' \tag{296}$$

where

$$\bar{M} = \sum_i \bar{M}_i = -\sum_i m_i R^2[b_i] \tag{297}$$

This shows that to update the moment of inertial we just need to pre and post-multiply a reference moment of inertial $\bar{M}$ times the current rotation matrix.

5.2 Coriolis Matrix

From the last section we saw

$$C_i = R'[L_i] \tag{298}$$

Thus

$$C = \sum_i C_i = \sum R'[L_i] = R'[\sum L_i] = R'[L] \tag{299}$$

Where the total angular momentum L is defined as follows

$$L = \sum L_i = \sum_i M_i \theta = M\theta = r\bar{M}r'\theta \tag{300}$$

Note since

$$C_i = R'[L_i] \tag{301}$$

is equivalent to

$$C_i \dot{\theta} = \theta \times L_i \tag{302}$$

then

$$C = R'[L] \tag{303}$$

is equivalent to

$$C\dot{\theta} = \theta \times L \tag{304}$$

5.3 Torques

The total torque is the sum of the torques applied to each particle

$$\tau = \sum_i \tau_i = \sum_i J' f_i = R[x_i] f_i = x_i \times f_i \tag{305}$$

Thus the equation of motion is as follows

$$\tau = M\ddot{\theta} + C\dot{\theta} \tag{306}$$

or equivalently

$$\tau = M\ddot{\theta} + \dot{\theta} \times L \tag{307}$$

with

$$\tau = \sum_i x_i \times f_i \tag{308}$$

$$M = r\bar{M}r' \tag{309}$$

$$\bar{M} = \sum_i R^2[b_i] \tag{310}$$

$$C = R'[L] \tag{311}$$

$$L = \bar{M}\dot{\theta} \tag{312}$$

5.4 Moments of Inertia: Parallel Axis Theorem

Tables exist for moments of inertia matrices of objects with prototypical shapes about the center of mass. In many cases we need to compute the moment of inertial with respect to rotation points other than the center of mass. The Parallel Axis Theorem tells us how to do so. Without loss of generality we choose the rotation point as the origin of our coordinates, i.e.,

$$x_i = rb_i \tag{313}$$

where b_i has the coordinates of the unrotated point i with respect to the rotation point, which is at the origin. Let $\bar{M}$ be the (zero rotation) moment of inertial about the origin

$$\bar{M} = -\sum_i m_i R^2[b_i] = \sum_i m_i \Big(\mathbf{1}(b_i' b_i) - b_i b_i'\Big) \tag{314}$$

Note

$$\sum_i m_i b_i b_i' = \sum_i (b_i - \bar{b} + \bar{b})(b_i - \bar{b} + \bar{b}) \tag{315}$$

$$= \sum_i m_i (b_i - \bar{b})(b_i - \bar{b})' + \sum_i m_i \bar{b}\bar{b}' + 2\bar{b} \sum_i m_i (b_i - \bar{b})' \tag{316}$$

$$= \sum_i m_i (b_i - \bar{b})(b_i - \bar{b})' + \bar{m}\bar{b}\bar{b}' \tag{317}$$

where $\bar{m} = \sum_i m_i$ is the total mass. The same technique can be used to show that

$$\sum_i m_i b_i' b_i = \sum_i m_i (b_i - \bar{b})'(b_i - \bar{b}) + \bar{m}\bar{b}'\bar{b} \tag{318}$$

Thus

$$M = \sum_i m_i \Big((b_i - \bar{b})'(b_i - \bar{b})\mathbf{1} - (b_i - \bar{b})(b_i - \bar{b})'\Big) \tag{319}$$

$$+ \bar{m}\Big(\bar{b}'\bar{b}\mathbf{1} - \bar{b}\bar{b}\Big) = -\sum_i m_i R^2[b_i - \bar{b}] - \bar{m}R^2[\bar{b}] \tag{320}$$

The Parallel Axis Theorem follows

$$\boxed{\bar{M} = \bar{M}_c + M_{\bar{b}}} \tag{321}$$

where M is the moment of inertia about the origin, which we chose to be the center of rotation,

$$\bar{M} = -\sum_i m_i R^2[b_i] \tag{322}$$

$\bar{M}_c$ is the moment of inertia about the center of mass

$$\bar{M}_c = -\sum_i m_i R^2[b_i - \bar{b}] \tag{323}$$

$\bar{m}$ is the total mass

$$\bar{m} = -\sum_i m_i \tag{324}$$

$\bar{b}$ is the center of mass

$$\bar{b} = \frac{\sum_i m_i b_i}{\bar{m}} \tag{325}$$

and $M_{\bar{b}}$ is the moment of inertia of a point with mass $\bar{m}$ located at the center of inertia

$$M_{\bar{b}} = -\bar{m}R^2[\bar{b}] \tag{326}$$

$$\tag{327}$$

Note: $\bar{M}_c$ is positive semidefinite , and $-R^2[b]$ is positive semidefinite so $\bar{M}$ is "bigger" than $\bar{M}_c$.

5.4.1 Example: Rectangular Prism

Calculate the moment of inertial about its point of rotation of an aluminum object in the shape of a rectangular box. The size of the rectangular prism in meters is $1 \times 0.04 \times 0.04$. *The density of aluminum is 2640 Kg per cubic meter. The axis of rotation is orthogonal to its length, and 1 centimeters from the end (see Figure 12.)*

First we go to tables and see that the moment of inertia of a rectangular box about its center of mass is as follows:

$$\bar{M}_c = \frac{\bar{m}}{12}\begin{bmatrix} y^2+z^2 & 0 & 0 \\ 0 & x^2+z^2 & 0 \\ 0 & 0 & x^2 y^2 \end{bmatrix} \tag{328}$$

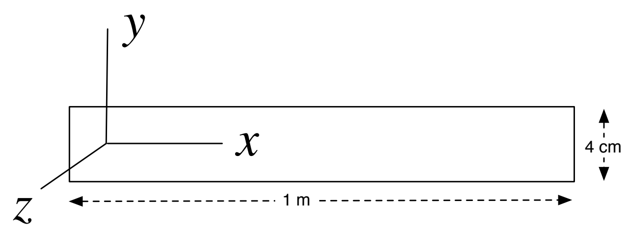

Figure 12: *Solid Pendulum.*

where $\bar{m}$ is the total mass of the object and x, y, z are its width, height and depth. In our case the mass is $1 \times 0.04 \times 0.04 \times 2640 = 4.224$ Kg. We let the long side of the box parallel to the x axis and the other two sides are parallel to the y and z axis respectively. Thus, the initial moment of inertia about the center of mass looks as follows

$$\bar{M}_c = \frac{4.224}{12} \begin{bmatrix} 0.0032 & 0 & 0 \\ 0 & 1.0016 & 0 \\ 0 & 0 & 1.0016 \end{bmatrix} = \begin{bmatrix} 0.0011 & 0 & 0 \\ 0 & 0.3526 & 0 \\ 0 & 0 & 0.3526 \end{bmatrix} \tag{329}$$

We put the origin at the center of rotation $(0.01, 0.02, 0.02)'$. Thus the coordinates of the center of mass are as follows follows

$$\bar{b} = \frac{1}{2}(1, 004, 0.04)' - (0.01, 0.02, 0.02)' = (0.49, 0, 0)' \tag{330}$$

Thus

$$-R^2[\bar{b}] = \bar{b}'\bar{b}\mathbf{1} - \bar{b}\bar{b}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.241 & 0 \\ 0 & 0 & 0.241 \end{bmatrix} \tag{331}$$

It follows that the zero rotation moment of inertia about the center of rotation is as follows

$$\bar{M} = \bar{M}_c - \bar{m}\ R^2[\bar{b}] = \begin{bmatrix} 0.0011 & 0 & 0 \\ 0 & 1.3706 & 0 \\ 0 & 0 & 1.3706 \end{bmatrix} \tag{332}$$

This tells us that the effective resistance to rotation about the y and z axis is equal and much larger than the resistance to rotation about the x axis.

5.5 Rotation about a fixed axis: Multiple Particles

In this case the axis of rotation u is fixed and

$$x_i = rb_i = e^{R[\theta u]} b_i \tag{333}$$

Thus

$$\begin{aligned} J_i &= R'[x_i]u & (334)\\ M_i &= m_i J_i' J_i = -m_i u' R^2[x_i]u = -m_i u' r R^2[b_i] r u & (335)\\ &= m_i \|u \times b_i\|^2 = -m_i u' R^2[b_i]u & (336)\\ M &= \sum M_i = u' = \sum_i m_i \|u \times b_i\|^2 & (337) \end{aligned}$$

Since each particle has zero Coriolis matrix, the total Coriolis matrix is also zero.

Regarding the total torque

$$\bar{\tau} = \sum_i J_i' f_i = \sum_i u' R[x_i] f_i = \sum_i u' \ (x_i \times f_i) \tag{338}$$

5.5.1 Case Study. Should Physics Textbooks Change the Way they Present Rotations About a Fixed Axis?

Consider a point with mass m rotating around the vertical axis as follows (see Figure 13. Left)

$$x = rb = e^{R[\theta]} b \tag{339}$$

where $\theta = (0, \lambda, 0)$ and $b = d\ (\sin\alpha, \cos\alpha, 0)'$ with α fixed. Most physics textbooks analyze this problem using the mathematics for the case in which θ is unconstrained. This means the only internal constraints are those that make sure the location of the pivot point (the origin) is fixed and the distance between the pivot and the rotating point is also fixed. In my opinion this is a didactic mistake: We say that the rotation is about a fixed axis of rotation but then we use the math for unconstrained axis. We will first analyze the problem treating θ as a 3 dimensional, unconstrained vector. Then we will analyze the problem the more natural way, i.e., treating the axis of rotation as having fixed orientation, in which case θ becomes a scalar. Note in this case

$$M = -mrR^2[b]r' \tag{340}$$

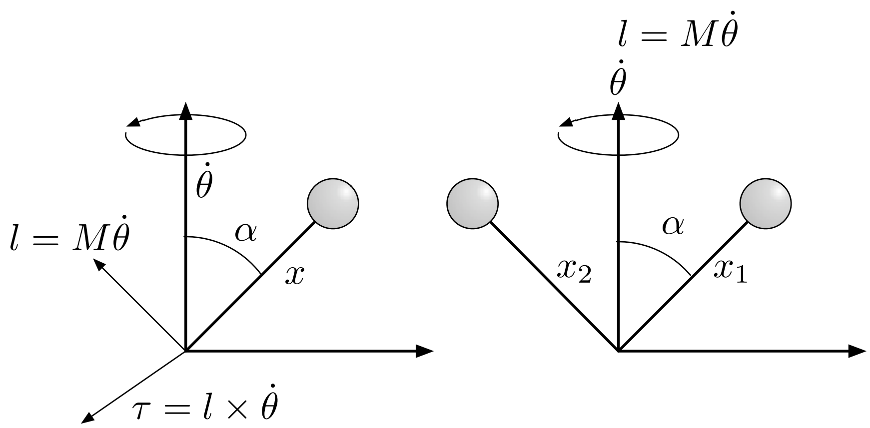

Figure 13: **Left:** *A point mass rotating about a fixed axis.* **Right:** *A balanced two point object rotating about a fixed axis.*

where

$$R^2[b] = d^2\mathbf{1} - d^2 \begin{bmatrix} \sin^2\alpha & \sin\alpha\cos\alpha & 0 \\ \sin\alpha\cos\alpha & \cos^2\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} = md^2 \begin{bmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha & 0 \\ -\sin\alpha\cos\alpha & \sin^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{341}$$

Thus the mass with respect to θ looks as follows

$$M = md^2 r \begin{bmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha & 0 \\ -\sin\alpha\cos\alpha & \sin^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} r' \tag{342}$$

The momentum with respect to θ looks as follows

$$l = M\dot{\theta} = md^2 r \begin{bmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha & 0 \\ -\sin\alpha\cos\alpha & \sin^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\lambda} \\ 0 \end{bmatrix} \tag{343}$$

$$= m\dot{\lambda}d^2 r \begin{bmatrix} -\sin\alpha\cos\alpha \\ \sin^2\alpha \\ 0 \end{bmatrix} \tag{344}$$

where we used the fact that $r\dot{\theta} = \dot{\theta}$. The Coriolis force on θ is as follows

$$C\dot{\theta} = \dot{\theta} \times l = m\dot{\lambda}d^2 \begin{bmatrix} 0 & 0 & \dot{\lambda} \\ 0 & 0 & 0 \\ -\dot{\lambda} & 0 & 0 \end{bmatrix} r \begin{bmatrix} -\sin\alpha\cos\alpha \\ \sin^2\alpha \\ 0 \end{bmatrix} \tag{345}$$

$$= m\dot{\lambda}^2 d^2 r \begin{bmatrix} 0 \\ 0 \\ \sin\alpha\cos\alpha \end{bmatrix} \tag{346}$$

Thus Newton's law in θ coordinates looks as follows

$$\tau = M\ddot{\theta}^2 + C\dot{\theta} = m\dot{\lambda}^2 d^2 r \begin{bmatrix} 0 \\ 0 \\ \sin\alpha\cos\alpha \end{bmatrix} \tag{347}$$

Thus, if we apply no external forces, i.e., $\tau = 0$ we get the following angular acceleration

$$\ddot{\theta} = -M^{-1}C\dot{\theta} = \dot{\lambda}^2 r \begin{bmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha & 0 \\ -\sin\alpha\cos\alpha & \sin^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \sin\alpha\cos\alpha \end{bmatrix} \tag{348}$$

We note that if $\alpha = \pi/2$, i.e, the point mass starts in the horizontal axis, then the Coriolis force is zero, and if no external force is applied, then the angular velocity is constant. The same occurs if $\alpha = 0$, i.e, the point mass starts in the vertical axis. However, for any other values of α there will be a Coriolis force on θ and thus the angular velocity will change through time in non-trivial ways. Figure 5.5.1 shows an example trajectory for an initial position $x = 2(\sin\pi/4, \cos\pi/4, 0)'$ an initial velocity $\dot{\theta} = (0, 1, 1)'$ and no external torque applied. Note how the y coordinate of the particle changes as a function of time, even though no external forces (including gravity) are present.

At first sight this seems strange. Why should the system behave so differently depending on the angle α ? After all α simply translates the point mass parallel to the axis of rotation. Why should a translation parallel to a fixed axis of rotation have such a large effect? The reason is that we are treating θ as a parameter with 3 degrees of freedom. So even though we say that the axis of rotation is fixed we use don't treat it as such. The

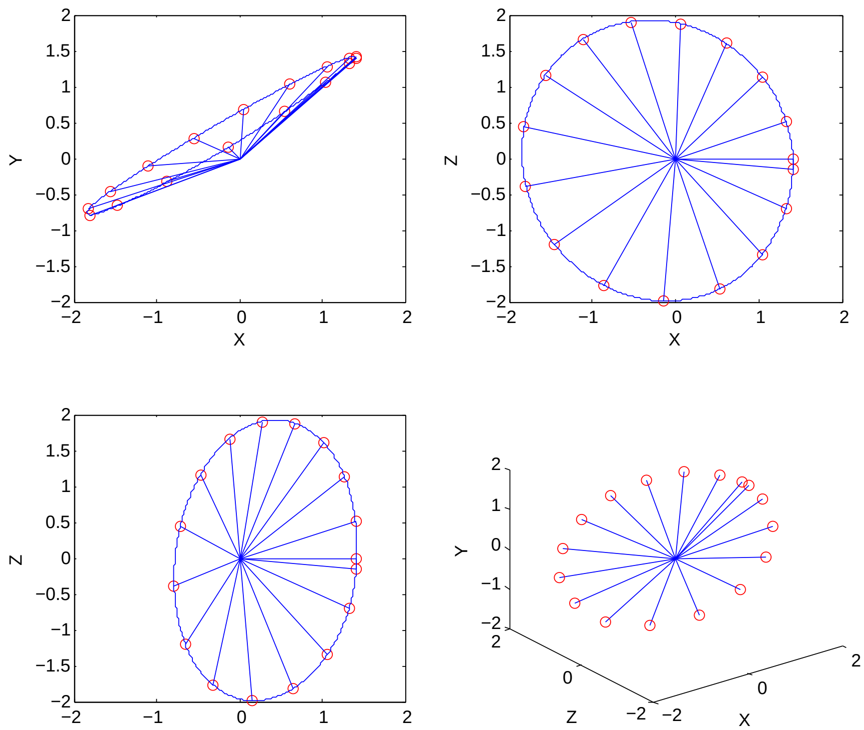

Figure 14: *Trajectory of a point mass starting at* $b = 2(\sin\pi/4, cos\pi.4, 0)'$ *with initial angular velocity* $(0, 1, 0)$ *and left to evolve with no external forces.*

only internal constraint we impose, is a force that fixes the distance of the point mass from the origin. Note the initial x velocity points in the direction of the negative side of the z axis. Thus the centrifugal force points in the direction of the horizontal axis. The component of this force parallel to x gets canceled by the internal constrain to maintain a constant distance from the origin. However the other component of the centrifugal force will push the mass downwards. On the other hand if we start the mass parallel to the horizontal axis, then the internal elastic force will cancel the centripetal force completely and thus the angular velocity will be constant. If we start the mass parallel to the vertical axis then the centrifugal force is zero and thus, the angular velocity will also be constant.

Now lets consider the case in the right side of Figure 13 with two masses located to the left and right of the axis of rotation. The total mass with respect to θ is the sum of the component masses, i.e., point masses

$$M = M_1 + M_2 = -md^2\Big(R^2[b_1] + R^2[b_2]\Big) \tag{349}$$

where $b_1 = (\sin\alpha, \cos\alpha, 0)'$ and $b_2 = (\sin(-\alpha), \cos(-\alpha, 0)$ Thus

$$M = 2md^2 \begin{bmatrix} \cos^2\alpha & 0 & 0 \\ 0 & \sin^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{350}$$

Thus, since the mass matrix is diagonal, the angular momentum $l = M\dot{\theta}$ is aligned with $\dot{\theta}$. Therefore the Coriolis force $C\dot{\theta} = \dot{\theta} \times l$ will be zero. As a consequence, in this case regardless of the value of α the angular velocity will be constant if no external forces are applied.

Now let's analyze the problem the way it should be done: treating the orientation of the axis of rotation as a fixed constraint, i.e.,

$$x = rb = e^{R[\theta u]}b \tag{351}$$

where θ is a scalar and $u = (0, 1, 0)'$. First let's consider the single point mass case. In this mass with respect to θ is as follows

$$M = -u'rR^2[b]r'u = -u'R^2[b]u = m(d\sin\alpha)^2 \tag{352}$$

The momentum with respect to theta is as follows

$$l = M\dot{\theta} = m\dot{\theta}(d\sin\alpha)^2 \tag{353}$$

which is constant as long as $\dot{\theta}$ is constant. Thus no Coriolis force is present and the mass will move with constant angular velocity regardless the value of α. The two point mass case behaves in a similar manner.

The key here is that if we say the axis of rotation is fixed, we shall treat all the forces responsible for maintaining it fixed as constraints. Unfortunately physics textbooks tend not to do so and as a consequence things become a bit mysterious when they should not be.

5.5.2 Example: Solid Pendulum

Formulate the equations of motion for the solid pendulum described in 5.4.1
In this case the axis of rotation u is the z axis. Thus

$$u' = [0, 0, 1] \tag{354}$$

$$M = u'\bar{M}u = u' \begin{bmatrix} 0.0011 & 0 & 0 \\ 0 & 1.3706 & 0 \\ 0 & 0 & 1.3706 \end{bmatrix} u = 1.3706 \tag{355}$$

Regarding the total torque produced by gravity we have

$$\bar{\tau} = \sum_i m_i u' R[r b_i] g = u' R[r \sum_i m_i \bar{b}]\ g = \bar{m} u' R[r\bar{b}]\ g \tag{356}$$

Note

$$r = e^{R[\theta u]} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{357}$$

where θ is the angle of rotation measured using the right hand side rule on the axis of rotation, i.e. the z axis. Thus rotation is measured with respect to deviation from the horizontal line. Positive rotations are clockwise. Note

$$r\bar{b} = r \begin{bmatrix} 0.49 \\ 0 \\ 0 \end{bmatrix} = 0.49 \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \tag{358}$$

$$R[r\bar{b}]\ g = 0.49 \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -9.81 \\ 0 \end{bmatrix} = 0.49 \begin{bmatrix} 0 \\ 0 \\ -9.81\cos\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -4.81\cos\theta \end{bmatrix} \tag{359}$$

and

$$\bar{\tau} = \bar{m} u' R[r\bar{b}]\ g = -(4.224)\ (4.81)\ \cos\theta = 20.32\ \cos\theta \tag{360}$$

Thus the equation of motion

$$\tau = M\ddot{\theta} \tag{361}$$

looks as follows

$$-20.32 \cos\theta = 1.3706\, \ddot{\theta} \tag{362}$$

Note the maximum torque is 20.32 Newton meters, i.e., 2878 Oz inches. i.e., we would need 7 HSR599TG Hitech servos to keep the pendulum in the horizontal position.

5.6 Principal Axis

We've seen that body rotating about an axis with no external torques applied to it will not have a constant angular velocity. This results from the Newtonian equations for rotations when there is no external torque τ

$$M\ddot{\theta} = -C\dot{\theta} \tag{363}$$

However there are special axes of rotation so that when an object spins about them it will continue to do so indefinitely with constant angular velocity. These are called the **principal axes** of the body. Thus θ to be a principal axis it must satisfy the following equation

$$C\dot{\theta} = 0 \tag{364}$$

where

$$C\dot{\theta} = J'\dot{J}\theta = \dot{\theta} \times l = \dot{\theta} \times (M\dot{\theta}) \tag{365}$$

Thus, for θ to be a principal axis, the momentum needs to be parallel to the velocity, i.e.,

$$M\dot{\theta} = \lambda\dot{\theta} \tag{366}$$

Note this requires for θ to be an eigenvector of M. We note that M is a positive semidefinite matrix, i.e., a covariance matrix, and thus the spectral theorem holds

Spectral Theorem: Any asymmetric positive semidefinite M can be decomposed as follows

$$M = p\Lambda p' \tag{367}$$

where p is an orthonormal matrix, whose columns are the principal axes of M (aka eigenvectors), and Λ is a diagonal matrix, whose entries are known as the eigenvalues of M.

5.6.1 Example: Principal Axis for a point mass.

Consider a single point mass located rotating about an axis θ

$$x = e^{R[\theta]} b \tag{368}$$

where $b = d^2(1,1,0)'$. Thus

$$M = r M_o r' \tag{369}$$

where

$$M_o = -mR^2[b] = md^2 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{370}$$

Using a standard package for finding eigenvectors we see that

$$M_o = md^2 \begin{bmatrix} -0.701 & -0.707 & 0 \\ -0.701 & 0.701 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -0.701 & -0.707 & 0 \\ -0.701 & 0.701 & 0 \\ 0 & 0 & 1 \end{bmatrix}' \tag{371}$$

Thus there are three axis we can rotate about with constant angular velocity while applying no torque: the z axis $(0,0,1)'$ the axis orthogonal to the initial location of the mass $(-1,1,0)'$, and the axis parallel to the location of the mass $(1,1,0)$, see Figure 15. Note, this analysis assumes the axis θ has three degrees of freedom. The analysis tells us that no torque would be needed to maintain the axis fixed in the three described above. This is because in this case the centrifugal force caused by the rotation is canceled by the internal constraints, that make sure the origin of the rotation is fixed and the distance from the origin to the point mass is also fixed. As explained above, if we treat the orientation of the axis as a fixed constraint, then we can rotate about any axis with constant angular velocity without having to apply any torque.

6 Rigid Bodies

A rigid body is a system of particles whose pairwise distances and angles are constant. Thus rigid objects satisfy the following constraints

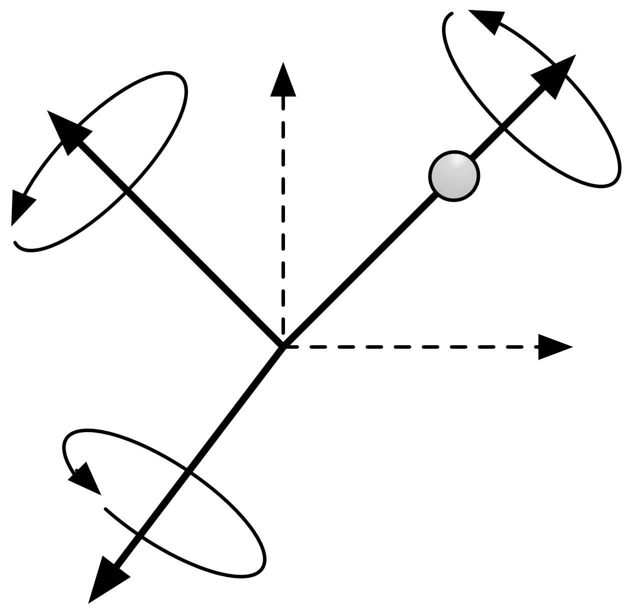

Figure 15: *The three principal axes of rotation for a point mass. A point mass is attached through a rigid rod to a pivot. The internal constraints fixed the distance to the pivot and the location of the pivot. There are the axis about which the point mass can rotate indefinitely at constant velocity without the need for applying a torque.*

$$x_i(t) = r(t)\, b_i + c(t) \tag{372}$$

To avoid clutter hereafter we omit the t argument. Note

$$\sum_i m_i x_i = r\left[\sum_i m_i b_i\right] + c\sum_i m_i \tag{373}$$

where m_i is the mass of particle i. Dividing by $\sum_i m_i$

$$\bar{x} = r\bar{b} + c \tag{374}$$

where

$$\bar{x} = \frac{\sum_i m_i x_i}{\sum_i m_i} \tag{375}$$

is the center of mass at time t and

$$\bar{b} = \frac{\sum_i m_i b_i}{\sum_i m_i} \tag{376}$$

Thus

$$x_i - \bar{x} = r b_i - r\bar{b} \tag{377}$$

or equivalently

$$x_i = \bar{x} + r(b_i - \bar{b}) \tag{378}$$

We parameterize r using the 3D vector α

$$r = e^{R[\alpha]} \tag{379}$$

and let $\theta = (\bar{x}', \alpha')'$

6.1 Total Moment of Inertia

Note

$$J_i = \frac{\partial x}{\partial \theta'} = \left[\begin{array}{cc} \mathbf{1} & R'[x_i - \bar{x}] \end{array} \right] \tag{380}$$

where $\mathbf{1}$ is the 3×3 identity matrix. In the last equation we used the fact that

$$\frac{\partial e^{R[\alpha]}(b_i - \bar{b})}{\partial \alpha'} = R'[r(b_i - \bar{b})] \tag{381}$$

Thus

$$M_i = m_i J_i' J_i = \left[\begin{array}{cc} \mathbf{1} & R'[x_i - \bar{x}] \\ R'[x_i - \bar{x}] & -R^2[x_i - \bar{x}] \end{array} \right] \tag{382}$$

and

$$M = \sum_i M_i = \left[\begin{array}{cc} \bar{m} & \mathbf{0} \\ \mathbf{0} & \bar{M} \end{array} \right] \tag{383}$$

where **0** is a 3×3 matrix of zeros and

$$\bar{m} = \sum_i m_i \tag{384}$$

$$\bar{M} = -\sum_i m_i R^2[x_i - \bar{x}] = -r\Big(\sum_i m_i R^2[b_i - \bar{b}]\Big)r' \tag{385}$$

In the last derivation we used the fact that

$$\sum_i m_i(x_i - \bar{x}) = 0 \tag{386}$$

6.2 Coriolis Matrix

Note

$$\dot{J}_i = \begin{bmatrix} \mathbf{0} & R'[\dot{x}_i - \dot{\bar{x}}] \end{bmatrix} \tag{387}$$

Thus

$$C_i = m_i J_i' \dot{J}_i = m_i \begin{bmatrix} 0 & R'[\dot{x}_i - \dot{\bar{x}}] \\ 0 & R'[L_i] \end{bmatrix} \tag{388}$$

where

$$L_i = -m_i R^2[x_i - \bar{x}]\dot{\alpha} \tag{389}$$

In the last step we used (259) which shows that

$$R'[L_i] = -R[x_i - \bar{x}]R[\dot{x}_i - \dot{\bar{x}}] \tag{390}$$

Thus

$$C = \sum_i C_i = \begin{bmatrix} 0 & 0 \\ 0 & \bar{C} \end{bmatrix} \tag{391}$$

where

$$\bar{C} = -\sum_i m_i R'[L_i] = R'[\bar{L}] \tag{392}$$

and

$$\bar{L} = \bar{M}\dot{\alpha} \tag{393}$$

In the last step we used the fact that $\sum_i m_i(x_i - \bar{x}) = 0$.

6.3 Torque

$$\tau_i = J_i' f_i = \begin{bmatrix} f_i \\ R[x_i - \bar{x}] f_i \end{bmatrix} = \begin{bmatrix} f_i \\ (x_i - \bar{x}) \times f_i \end{bmatrix} \tag{394}$$

Thus

$$\tau = \sum_i \tau_i = \begin{bmatrix} \bar{f} \\ \bar{\tau} \end{bmatrix} \tag{395}$$

where

$$\bar{f} = \sum_i f_i \tag{396}$$

$$\bar{\tau} = \sum_i (x_i - \bar{x}) \times f_i \tag{397}$$

6.4 Euler Equations of Motion for Rigid Bodies

Thus the equations of motion in θ coordinates look as follows

$$\begin{bmatrix} \bar{f} \\ \bar{\tau} \end{bmatrix} = \begin{bmatrix} \bar{m} & 0 \\ 0 & \bar{M} \end{bmatrix} \begin{bmatrix} \ddot{\bar{x}} \\ \ddot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \bar{C} \end{bmatrix} \begin{bmatrix} \dot{\bar{x}} \\ \dot{\alpha} \end{bmatrix} \tag{398}$$

Note the equations of motion decompose into a translational component, commonly known as ***Euler's First Law of Motion***

$$\boxed{\bar{f} = \bar{m}\,\ddot{\bar{x}}} \tag{399}$$

and a rotational component commonly known as ***Euler's Second Law of Motion***.

$$\boxed{\bar{\tau} = \bar{M}\ddot{\alpha} + \bar{C}\dot{\alpha}} \tag{400}$$

where

$$\bar{M} = -r\Big(\sum_i m_i R^2[b_i - \bar{b}]\Big)r' \tag{401}$$

$$\bar{L} = M\dot{\alpha} \tag{402}$$

$$\bar{C} = R'[\bar{L}] \tag{403}$$

$$\tau = \sum_i (x_i - \bar{x}) \times f_i \tag{404}$$

The first Law tells us that if the sum of all the external forces is zero, the particles may rotate around but the center of gravity will not accelerate. The equation for rotational motion is It tells us that if the sum of the torques is zero, the center of gravity may accelerate but the rotation about the center of gravity will have constant angular velocity ω.

6.5 Computer Simulation of Rigid Motion Dynamics (needs revision)

There are three reference frames of interest: (1) World frame, an inertial frame of reference. (2) Center of mass frame Axis parallel to the world coordinates, origin at center of mass. (3) Body frame. Coordinates of the body particles do not vary with time. The state of all the particles in the object can be condensed into the location of the center $\bar{x}(t)$ of mass and the rotation of the body frame of reference with respect to the world coordinates.

Initialization:

- Specify the mass of the particles $\{m_1, \cdots, m_n\}$.
- Compute the total mass

$$\bar{m} = \sum_k m_k \tag{405}$$

- Specify the geometry of the model, i.e., position of the particles in body coordinates $\{b_1, \cdots, b_n\}$ so that the center of mass is at the origin, i.e.,

$$\frac{\sum_k b_k m_k}{\sum_k m_k} = 0 \tag{406}$$

- Compute the initial moment of inertia matrix

$$I_b \stackrel{\text{def}}{=} I - o(\bar{x}_o) = \sum_k m_k \Big(b'_k b_k \mathbf{1} - b_k b'_k \Big) \tag{407}$$

- Specify the initial location of the center of mass $\bar{x}_o$ and the initial rotation vector u_o.

- Transform the rotation vector into a rotation matrix

- Specify the initial translational and rotational velocities: $\bar{v}_o, du_o\ dt$.

- Specify the initial forces, in world coordinates, applied to all the particles: $f_o(1), \cdots, f_(n)$.

- Choose a type step $\Delta_t > 0$.

Translation Update Equations

- Compute the total force in world coordinates

$$\bar{f}_t = \sum_{i=1}^{n} f_t(i) \tag{408}$$

- Apply Euler's first law to obtain the acceleration of the center of mass

$$\bar{a}_t = \bar{f}_t / \bar{m} \tag{409}$$

- Update the velocity and location of the center of mass by assuming constant acceleration throughout the interval $[t, \Delta_t]$

$$\bar{v}_{t+\Delta_t} = \bar{v}_t + \bar{a}_t \Delta_t \tag{410}$$

$$\bar{x}_{t+\Delta_t} = \bar{x}_t + \bar{v}_t \Delta_t + \frac{\Delta_t^2}{2} \bar{a}_t \tag{411}$$

Rotation Update Equations

- Compute the rotation matrix

$$r_t = e^{R[u_t]} \tag{412}$$

- Compute the location of the particles in world coordinates

$$x_t(k) = r_t\, b_k + \bar{x}_t \tag{413}$$

- Compute the total torque in body centered and body oriented coordinates

$$\tau_t(\bar{x}(t), k) = (x_t(k) - \bar{x}_t) \times f_t(k) \tag{414}$$

$$\bar{\tau}_t(\bar{x}_t) = \sum_{k=1}^{n} \tau_t(\bar{x}_t, k) \tag{415}$$

$$\tilde{\tau}_t = r_t' \bar{\tau}_t(\bar{x}_t) \tag{416}$$

- Use Euler's second law to compute the total angular acceleration with respect to the center of mass:

$$\frac{d^2 u_t}{dt^2} = I_b^{-1}\big(\tilde{\tau}_t - \frac{du_t}{dt} \times I_b \frac{du_t}{dt}\big) \tag{417}$$

- Update the angular velocity and orientation vector by assuming constant angular acceleration across the $[t, t + \Delta_t]$ interval

$$\frac{du_{t+\Delta t}}{dt} = \frac{du_t}{dt} + \Delta_t \frac{d^2 u_t}{dt^2} \tag{418}$$

$$u(t + \Delta t) = u(t) + \Delta_t \frac{du_t}{dt} + \frac{\Delta_t^2}{2} \frac{d^2 u_t}{dt^2} \tag{419}$$

7 Articulated Bodies: Kinematic Links (Needs Revision)

7.1 Inverse Kinematics

Error function

$$\rho = \frac{1}{2}\|d - f(x)\|^2 \tag{420}$$

where $d \in \mathcal{R}^3$ is the desired location in spaced, $x \in \mathcal{R}^n$ are kinematic control variables, e.g., joint angles.

Jacobian Transpose Method This method can be derived as the gradient descent rule for minimizing the error function. Using the chain rule (seem matrix calculus tutorial) we have

$$\nabla_x \rho = \nabla_x f(x) \nabla_{f(x)} \rho = J_f(x)^T (d - f(x)) \tag{421}$$

Moving in the direction of the gradient we get the Jacobian transpose rule commonly used to solve inverse kinematics problems

$$x(t+1) = x(t) + \kappa J_f(x)^T (d - f(x)) \tag{422}$$

where κ is a small positive constant.

Jacobian Inverse Method This method can be derived as the Gauss-Newton rule for minimizing the error function. At each iteration we linearize f about the current state of the variables

$$\hat{\rho}(x) = \frac{1}{2}\|d - \hat{f}(x)\|^2 = \frac{1}{2}\|d - (f(u) + J_f(u)(x-u))\|^2 = \frac{1}{2}\|(d - f(u) + J_f(u)u) - J_f(u)x\|^2 \tag{423}$$

where u is the fixed state about which we make the approximation. Minimizing $\hat{\rho}$ with respect to u is a linear regression problem of well known solution:

$$\hat{x} = \left[J_f(u)^T J_f(u)\right]^{-1} J_f(u)^T (d - f(u) + J_f(u)u) \tag{424}$$

$$= u + \left[J_f(u)^T J_f(u)\right]^{-1} J_f(u)^T (d - f(u)) \tag{425}$$

This suggests the following update rule

$$x(t+1) = x(t) + \left[J_f(x(t))^T J_f(x(t))\right]^{-1} J_f(x(t))^T (d - f(x(t))) \tag{426}$$

An interesting property of this method is that it produces straight lines from the initial point to the desired point.

From a Newton-Raphson optimization perspective, the term $\left[J_f(u)^T J_f(u)\right]^{-1}$ can be seen as an approximation to the inverse Hessian. Gradient descent can also be seen as Newton-Raphson approximating the Hessian by the identity matrix.

Adding a Ridge Term I have not seen this method around but I found it to be much more stable than the standard inverse Jacobian method. We add a penalty term for large values of x (may be better to modify it to penalize large changes in x rather than large x).

$$\rho(x) = \frac{1}{2}\|d - f(x)\|^2 + \alpha x^T x \tag{427}$$

and apply Gauss-Newton.

$$\hat{\rho}(x) = \frac{1}{2}\|d - f(u) - J_f(u)(x-u)\|^2 + \alpha x^T x \tag{428}$$

and apply Gauss-Newton. Local minimization of rho is a linear ridge regression problem with well known solution

$$\hat{x} = u + \left[J_f(u)^T J_f(u) + \alpha I_n\right]^{-1} J_f(u)^T (d - f(u)) \tag{429}$$

Suggesting the following update rule

$$x(t+1) = x(t) + \left[J_f(x(t))^T J_f(x(t)) + \alpha I_n\right]^{-1} J_f(x(t))^T (d - f(x(t))) \tag{430}$$

7.2 Example 2D Kinematic Chain

Let $i = 1, m$ index the links. Let $x = (x_1, \cdots, x_m)^T, \alpha = (\alpha_1, \cdots, \alpha_m)$ where $x_i \in \mathcal{R}^2$, $\alpha_i \in \mathcal{R}$, $\_i \in \mathcal{R}$ are the position, angle and length of link i (see Figure 16)

Direct Kinematics

$$x_i = x_{i-1} + \begin{bmatrix} \cos(\sum_{j=1}^{i} \alpha_i) \\ \sin(\sum_{j=1}^{i} \alpha_i) \end{bmatrix} \tag{431}$$

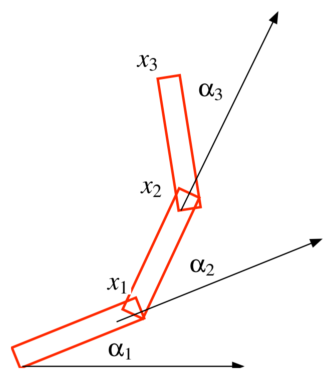

Figure 16: *A simple kinematic chain.*

Inverse Kinematics Suppose we want to find a set of joint angles that position the tip of the kinematic chain at a desired location d. Our error function is

$$\rho(\alpha) = \frac{1}{2}\|d - f(\alpha)\|^2 \tag{432}$$

where

$$f(\alpha) = x_m(\alpha) \tag{433}$$

Thus

$$\frac{\partial f(\alpha)}{\alpha_k} = \sum_{i=k}^{m} \left[\begin{array}{c} -\sin(\sum_{j=1}^{k} \alpha_i) \\ \cos(\sum_{j=1}^{k} \alpha_i) \end{array} \right] \tag{434}$$

from which the Jacobian can be easily constructed. This tutorial includes an openGL example of this case.

8 Appendix

8.1 Weighted Least Squares

Let $y = xb + e$, where $y, e \in \mathcal{R}^n, b \in \mathcal{R}^p$ and x is a $n \times p$ matrix. y, x are known and the goal is to find a vector $\hat{b}$ that minimizes

$$\rho(b) = \frac{1}{2}(y - xb)'w(y - xb) \tag{435}$$

i.e.,

$$\hat{b} = \underset{b}{\operatorname{argmax}}\, \rho(b) \tag{436}$$

where w is a symmetric positive definite matrix. To get the minimum of ρ we take the gradient and set it to zero

$$\nabla_b \rho = \nabla_b (y - xb) \nabla_{xb} \rho = -x'w(y - xb) = 0 \tag{437}$$

Thus at the minimum

$$x'wxb = x'wy \tag{438}$$

and therefore

$$\hat{b} = (x'wx)^{-1} x'wy \tag{439}$$

We let $\hat{y}$ $\hat{e}$ represent the estimates of y and x, i.e.,

$$\hat{y} = x\hat{b} \tag{440}$$

$$\hat{e} = y - \hat{y} \tag{441}$$

Note

$$\hat{y} = x\hat{b} = hy\hat{e} = y - \hat{y} = (I - h)y \tag{442}$$

where

$$h = x(x'wx)^{-1} x'w \tag{443}$$

Thus

$$\hat{y}'w\hat{e} = y'h'w(I - h)y = y'h'wy - y'h'why = 0 \tag{444}$$

where we used the fact that

8.1.1 Geometric Interpretation

The matrix w induces an inner product and a norm in $\mathcal{R}^p$

$$< x, y >= x'wx \tag{445}$$

$$|x|^2 =< x, x >= x'wx \tag{446}$$

Our goal is to find a value $\hat{b}$ of b such that $\hat{y} = x\hat{b}$ is as close as possible to y with respect to the w-based norm. This is achieved by having $y = \hat{y} + \hat{e}$ with $\hat{y}$ being w-orthogonal to $\hat{e}$.

8.2 Friction and Drag

Coulomb Friction: Consider a block of mass m sliding along a horizontal surface. The static coefficient of friction tells us the force F needed to start moving the solid

$$F = \mu W \tag{447}$$

where μ is the static friction coefficient, and W is the force perpendicular to the direction of motion. In the case of gravitational force $W = mg$. As the solid starts to move, the frictional resistance changes and is called dynamic friction coefficient. Typically it is smaller than the static friction coefficient and relatively constant regardless of the speed.

Viscous Friction: As such, friction is different from *drag* (aka viscous friction) which is the force opposing motion of a solid in a fluid (gas or liquid). Drag is due to the viscosity of the fluid and, contrary to friction, increases with the velocity of the object.

The drag equation is due to Rayleigh

$$F = -\frac{1}{2}\rho ac\|v\|^2 u \tag{448}$$

where F is the drag force, ρ is the fluid's viscosity, a is the effective (or reference) area, c is the drag coefficient, and u is a unit vector parallel to the velocity vector v.

It is useful to think of ac as a constant that characterizes a specific object (e.g., a car). For example for a Honda Insight 1999 car it is 0.47 m^2 (squared meters).

The power to overcome the drag force at a given velocity is

$$P = f \cdot v = \frac{1}{2}\rho v^3 ac \tag{449}$$

which increases with the cube of the velocity.

At low velocities, the drag force can be approximated by Stokes law

$$F = -bv \tag{450}$$

For small spheres, Stokes showed that

$$b = 6\pi\rho r \tag{451}$$

where ρ is the fluid viscosity and r is the radius of the sphere. A common unit of viscosity is the $dynessec/cm^2$ or Poise. Another unit is the Pascal second Pa s, which equals 10 Poises. Here are some useful viscosity coefficients

Glicerine at 40 degrees Centigrade:	1420 milli Pas.
Water at 40 degrees Centigrade:	0.65 milli Pas
Corn Oil at 20 degrees Centigrade:	65 milli Pas
Honey at 20 degrees centigrade:	10 Pas
Air at 15 degrees centigrade:	17.9 micro Pas

Note under Stoke's drag model, given a constant force f we have

$$\frac{dv_t}{dt} = \frac{-1}{bm}v_t + \frac{f}{m} \tag{452}$$

thus

$$v_t = v_\infty + (v_0 - v_\infty)e^{-bt/m} \tag{453}$$

where

$$v_\infty = \frac{f}{b} \tag{454}$$

Thus, the equilibrium velocity is proportional to the force and the relaxation time constant is m/b.

If the object is a sphere, we get

$$f = mg = \frac{4}{3}\pi r^3 \eta g \tag{455}$$

$$b = 6\pi\rho r \tag{456}$$

$$v_\infty = \frac{4}{18} g \frac{\eta}{\rho} r^2 \tag{457}$$

where η is the density of the sphere's material and g is gravity's constant. Thus, under this model, a sphere with twice the diameter should have 4 times terminal velocity. This corresponds to the intuition that a larger sphere of iron will fall faster than a smaller sphere of iron. Also, for a given radius, a sphere of a denser material will fall faster than a sphere of a less dense material (aha, Galileo was wrong after all).

8.3 Units of Force and Rotational Torque

- A **Newton** is the force needed to accelerate one Kg of Mass 1 $meter/sec^2$. On Earth 1 Kg of Mass exerts a force of 9.81 Newtons. On Earth you can think of a Newton as approximately 100 grams (3.5 ounzes, 0.22 pounds).
- A **Dyne** is the force needed to accelerate one gram of Mass 1 cm/sec^2. One dyne euquals 10^{-5} Newtons.
- A **Kilopond** or kilogram force. The force exerted by standard gravity on a kilogram of mass. 9.81 Newtons equals 1 Kilopond.
- **Newton Meter**: Torque produced by 1 Newton applied one meter from axis of rotation. Think of it as torque produced by 100 grams 1 meter from the origin, or about 1 Kilo 0.1 meter from the origin.
- **Newton Milli Meter**: Or Milli Newton Meter. Torque produced by 1 Newton applied one Milli meter from axis of rotation. Think of it as torque produced by 100 grams 1 milliner meter from the origin.
- **Dyne centimeter**: torque produced by 1 dyne one cm from axis of rotation: 10^{-7} Newton meters.
- **Ounce inch**: 1 Newton Meter = 141.612 Oz inches
- **Kg cm**: 1/10.197 Newton meters.
- Hitec HSR5995TG servo has a max torque of 416 Oz Inches, i.e., 2.94 Newton Meters.

9 Appendix

SI Units (International System of Units)

- Mass: Kg
- Force: Newton
- Pressure: Pascal ($Newton/m^2$)
- Power: Watt

- Energy: Joule (Newton Meter)
- Electric Potential: Volt
- Charge: Columb
- Capacitancd: Farad
- Resistance Ohm
- Inductance: henry
- Length: meter
- Current: Ampere
- Time: Second
- Torque (moment of force): Newton Meter
- Moment of Inertia: Kg m^2
- Angular Velocity: Radians/Sec