Linked list with loop

Jacob Whitehill
jake@mplab.ucsd.edu

Let positive integers s_1 and s_2 denote the "speeds" of Iterator 1 and 2, respectively, where speed is the number of nodes that each iterator traverses in one timestep. Let non-negative integer l denote the number of nodes in the linked list *prior to* the start of the loop. Finally, let positive integer k denote the number of nodes *in* the loop.

Here we show that the two iterators must "intersect" at the same node inside the loop after a finite number of timesteps. An *intersection* is defined to occur when the "distance" of Iterator 1 from the start of the loop is equal to the "distance" of Iterator 2 from the start of the loop *at the same moment in time.* The distance is a number in $\{0, \dots, k-1\}$ that specifies which node in the loop (starting at 0) the iterator currently points to. Let m be the smallest non-negative integer such that $ms_1 \geq l$ and $ms_2 \geq l$. In other words, m is the number of timesteps that Iterator 1 and 2 must iterate so that they have *both* entered the loop. Then $ms_1 - l \mod k$ (or $ms_2 - l \mod k$) is the distance of Iterator 1 (or Iterator 2) from the start of the loop after m timesteps.

We wish to find a number of *additional* timesteps $t \geq 0$ (after the initial m timesteps) such that the two iterators intersect in the loop. More precisely, we wish to find a t that satisfies the congruency

$$
\begin{aligned}
ms_1 - l + ts_1 &\equiv ms_2 - l + ts_2 \mod k && (1)\\
m(s_1 - s_2) + t(s_1 - s_2) &\equiv 0 \mod k && (2)\\
(m + t)(s_1 - s_2) &\equiv 0 \mod k && (3)
\end{aligned}
$$

If $s_1 = s_2$ (i.e., the iterators travel at the same speed), then this congruency is satisfied for all values of t. Otherwise ($s_1 \neq s_2$), this congruency is satisfied whenever $m + t = nk$ for any integer (of our choosing) $n \geq 0$ because then $(m + t)(s_1 - s_2)$ is an exact multiple of k. In order to guarantee non-negativity of t, we choose n to be the smallest integer such that $nk \geq m$. Then, a suitable t can be found by solving

$$
\begin{aligned}
m + t &= nk && (4)\\
t &= nk - m && (5)
\end{aligned}
$$

Since we have found a $t \geq 0$ at which both iterators intersect, and since t was the number of *additional* timesteps after the initial m timesteps required to *enter* the loop, we conclude that the two iterators will intersect after a total of $t + m = nk - m + m = nk$ timesteps, where n is the smallest non-negative integer such that $nk - m \geq 0$.