CSE 12:
Basic data structures and object-oriented design

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Lecture Eleven
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Main points from last lecture.

Exploiting relations over data

- Simple data structures such as lists store data without regard to order relations between elements.
- Lists offer *O*(1) add, but *O*(*n*) find and remove operations.
- In many applications, the user will want to *find* more often than *add*.
    - Even though a user may have "partial knowledge" (*key*) of an object, it may need the `find(o)` method to obtain the "whole record" (*value*).

Exploiting relations over data

- In Java, binary order relations can be defined between pairs of elements using the **`Comparable<T>`** interface, which includes the **`int compareTo (T o)`** method.
- Exploiting order relations enables us to achieve superior asymptotic time costs for find/removal operations.
- One prominent example of "accelerating search" is the binary search algorithm:
  - Assumes input array is already sorted.
  - Achieves $\log_2(n)$ worst-case time cost by *recursively* dividing the list into two halves and searching only the relevant half.

Recursion

- Recursion is a tool for defining mathematical structures and constructing algorithms.
- Every recursive algorithm/definition contains:
  - A self-referential *recursive part*; and
  - A *base case* to prevent circularity.
- Recursive definitions can be specified in formal languages like Backus-Naur Form (BNF) to facilitate *automatic generation of code* (e.g., "compiler compilers").

Recursion

- Recursion is ubiquitous in computer science:
  - Binary search executes recursively.
  - The input array to binary search is typically sorted using (recursive) MergeSort or QuickSort.
  - Data structures such as trees, and even linked lists, can be formulated recursively.
    - E.g., “a linked list is either a node, or a node followed by a linked list.”
  - Source code itself is a recursive structure.
  - Compilers “lex” and “parse” the individual symbols/tokens of source code using algorithms generated automatically from recursive definitions of source code structure.

Binary search

- Binary search is a recursive algorithm for finding a target value in a sorted array of data in $\log_2(n)$ (worst-case) time.
- Binary search requires a binary order relation to be defined on all the elements.
- For primitive numeric types like int, double, etc., we can use > or >= to compare data.
- For objects, we can use the `int compareTo(T o)` method of the `Comparable<T>` interface.

Recursive data structures

- Despite the $\log_2(n)$ efficiency of binary search, its utility on *lists* is limited:
    - Binary search would be very efficient on a linked list because of the lack of ability to “jump” to an arbitrary node.
    - Binary search on array-based lists is efficient; however, the array must be maintained in *sorted order*.
        - If the user adds a new element, the “correct spot” must be located and all subsequent elements “shifted over”.

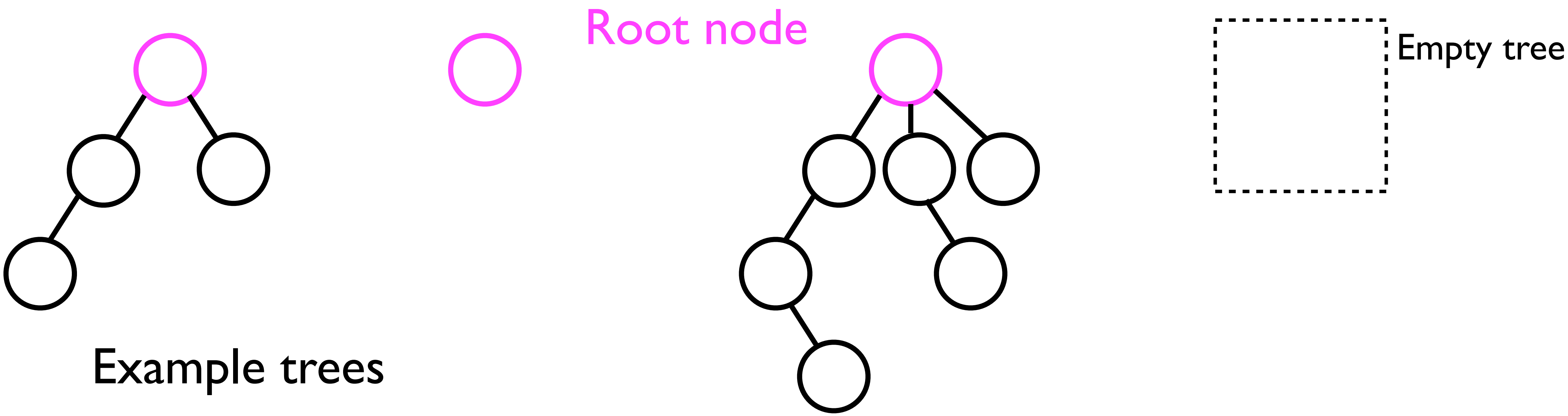
Recursive data structures

- It would be desirable to create data structure that offer efficient implementations of both **`add(o)`** and **`find(o)`**/**`remove(o)`**.
- Over the next few lectures we will cover two such structures -- heaps and binary search trees.
- Both these ADTs are based on *binary trees*, which are non-linear recursive data structures.

Binary Trees

Trees

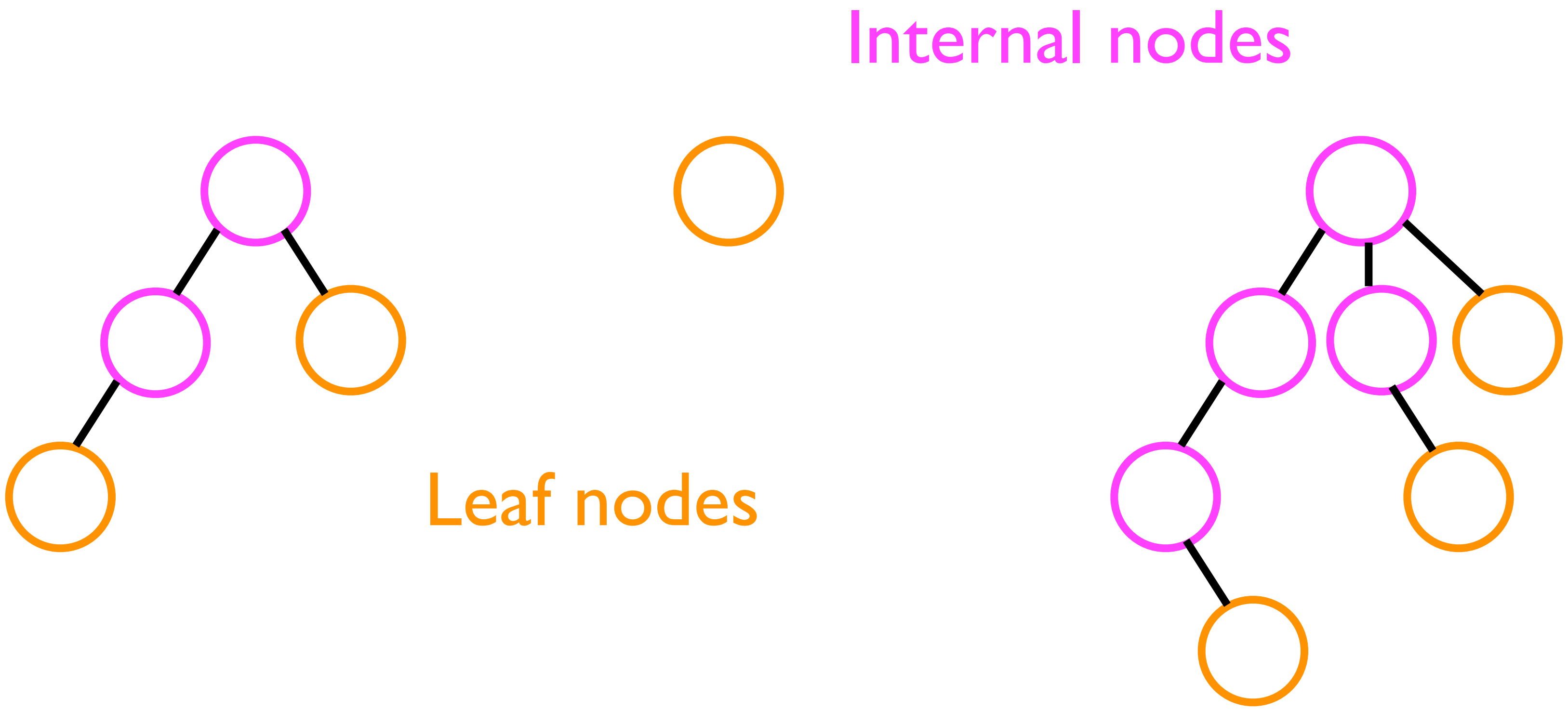
- A *tree* is an interconnected set of *nodes* that are organized in a hierarchy.
  - There is one node labeled the *root* of the tree.
  - Every node except the root has exactly 1 *parent* node.
  - Each node may have 0 or more *child* nodes (“children”).
    - Cycles are prohibited -- only one path may exist between any pair of nodes.
  - Parents and children are connected by *edges*.



Example trees

Trees

- A node with no children is called a *leaf*.
- A node with at least one child is called an *internal node*.



Internal nodes

Leaf nodes

Depth, height, and level

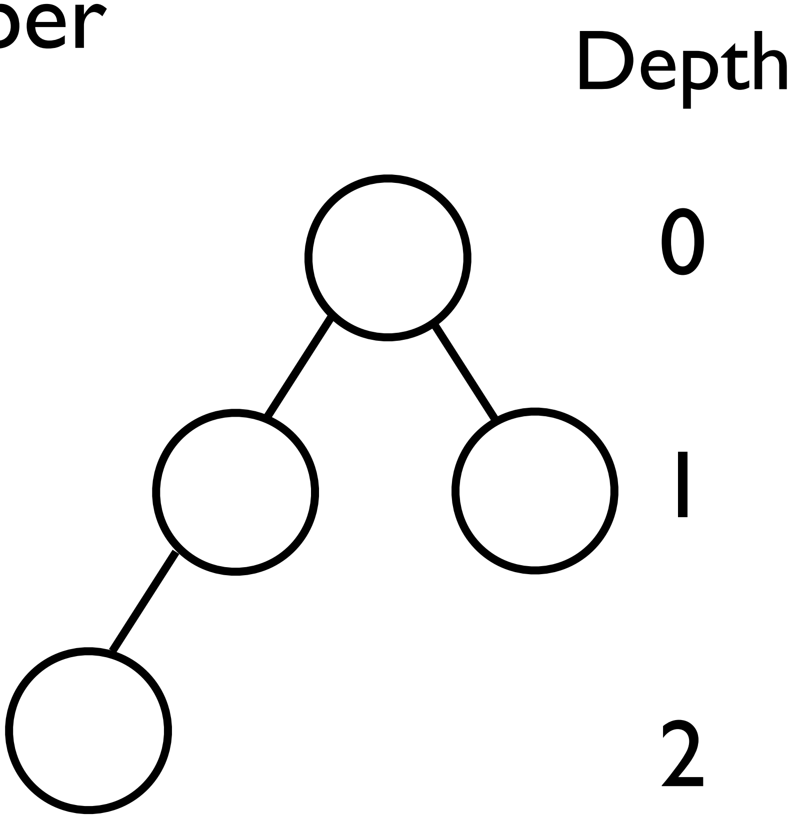
- Depth (iterative definition):
  - The *depth* of a node *N* is the number of edges between *N* and the root.
  - The root has depth 0.
- Depth (recursive definition):

Base case

  - The depth of a node *n* is 0 for the root; or

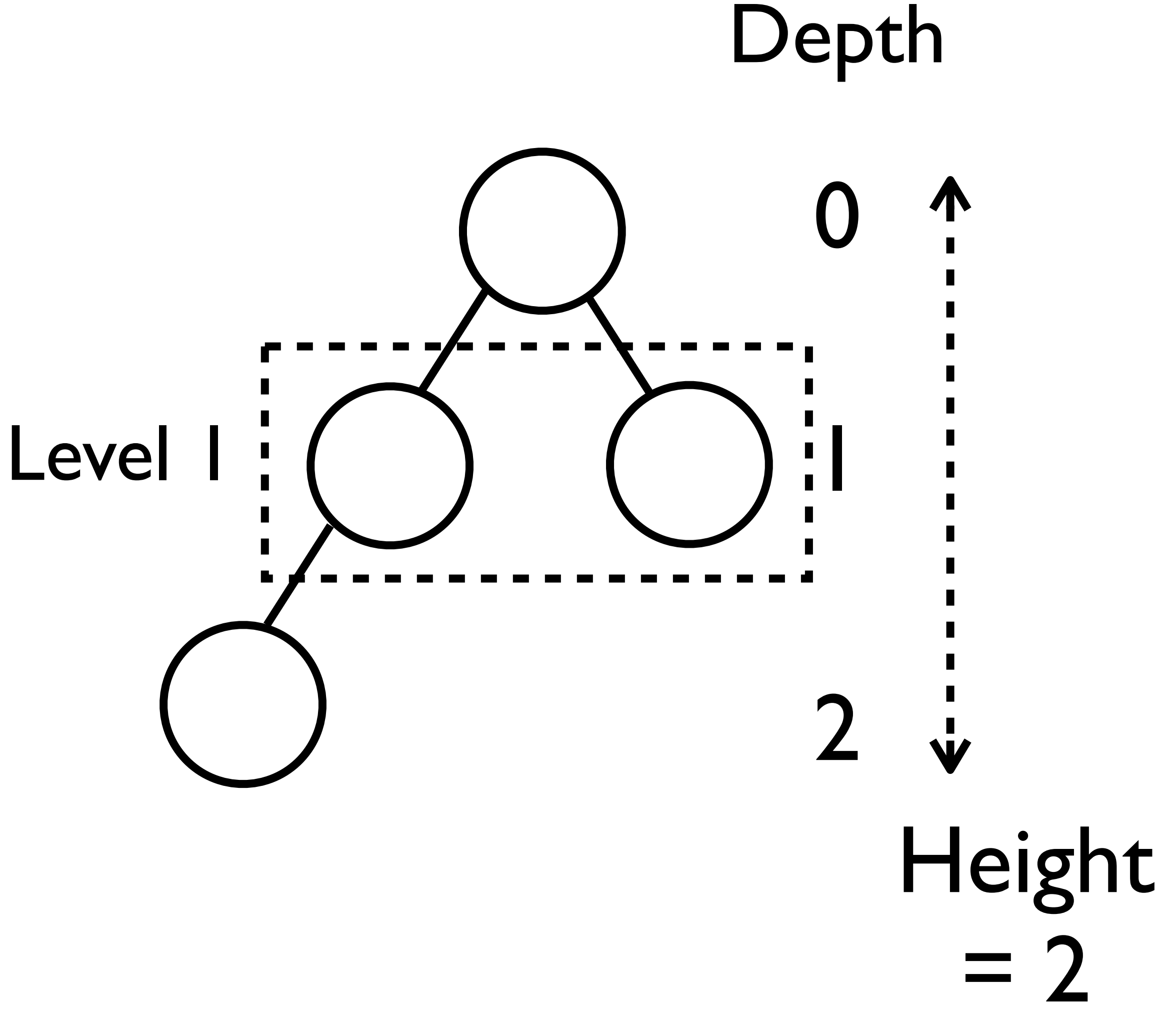
Recursive part

  - 1 + the depth of *n* parent node.



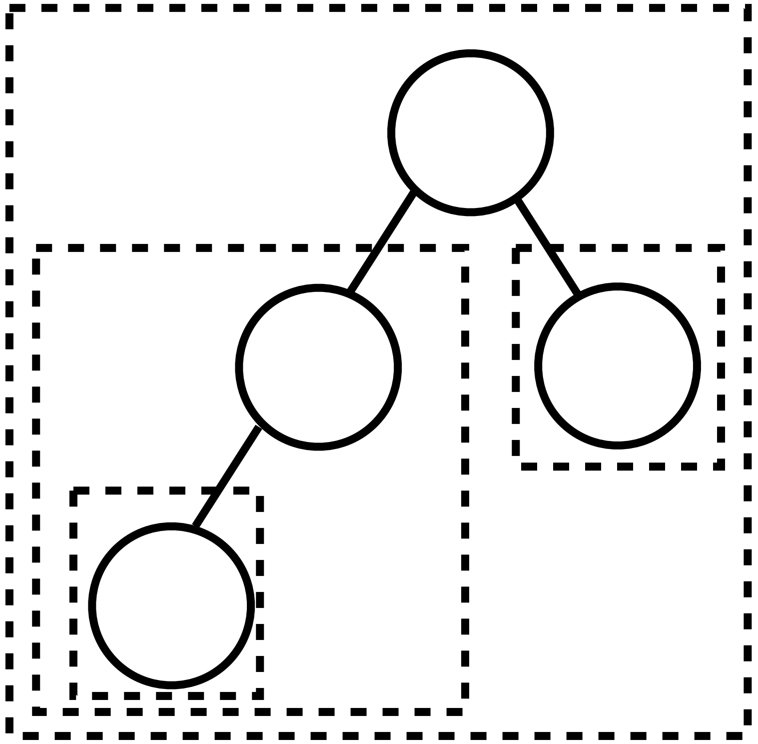
Depth, height, and level

- The *height* of a tree *T* is the maximum depth of any node in the tree.
  - Equivalent to length of longest path from the root to any leaf.
- A *level* of the tree consists of all the nodes at a particular depth.



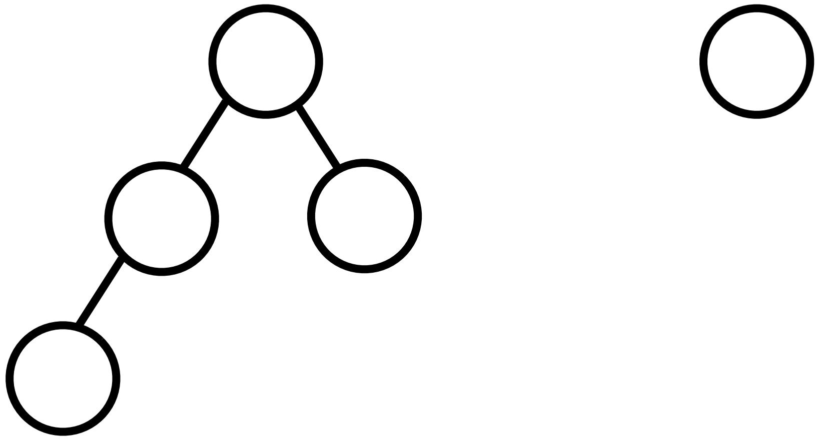
Sub-trees

- Each node in a tree is the *root* of its own *sub-tree*.
- The gray boxes below show all possible sub-trees.

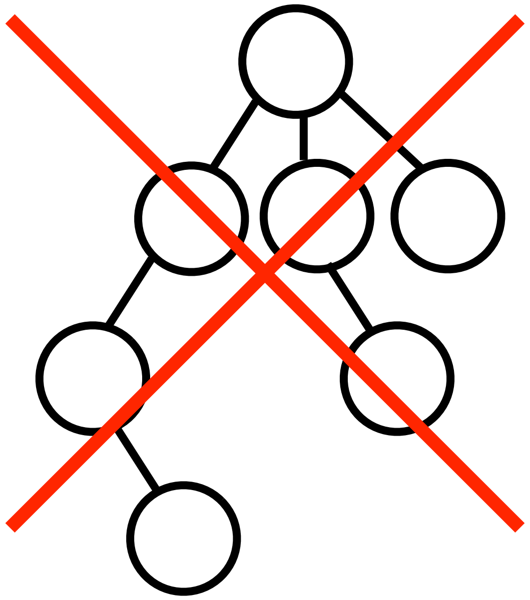


Binary trees

- A *binary tree* is a tree in which every node has at most 2 children.



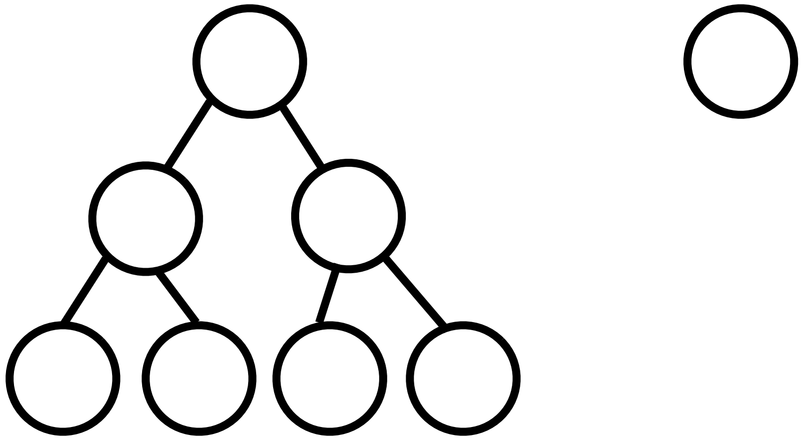
Examples of binary trees



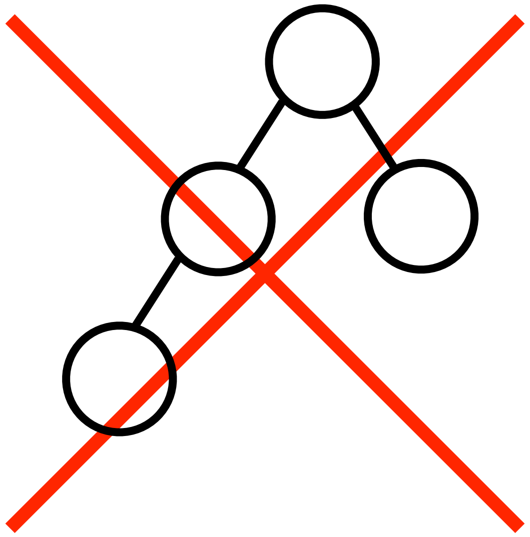
Not a binary tree

Binary tree properties

- A binary tree of height *h* is *full* if every node at depth *d* < *h* has 2 children.



Examples of full binary trees



Not a full binary tree

Binary tree properties

- A full binary tree with height h has 2^h leaf nodes and $2^{h+1} - 1$ nodes in total.
- Conversely, a full binary tree with n nodes total has height $\log_2(n+1)-1$.

Binary tree properties

- More generally, a binary tree *T* (not necessarily full) with *n* nodes has:
  - Minimum height $\log_2(n+1) - 1$ (when *T* is full).
  - Maximum height $n-1$ (when *T* is just a "chain" of nodes in which no node has more than 1 child).
- Why important?
  - The time cost of important tree operations such as `find(o)` depend on the average/maximum height of an arbitrary node in the tree.

Tree nodes

- Like nodes in a linked list, nodes in a tree contain a *data element* (otherwise, trees would be useless for ADTs).
- However, nodes in a tree contain more than 2 “links” (edges) to other nodes.
  - One link to parent node.
  - One link to each child node.

Node class for general trees

- From this description, we can create a **`Node`** class for use in *general* trees:

```
class Node<T> {
  Node<T> _parent;  // link to parent node
  Node<T>[] _children;  // links to children
  int _numChildren;
  T _data;  // data element the node stores
}
```

- Alternatively, we can used a *linked list* to manage the child **`Node`**s:

```
class Node<T> {
  Node<T> _parent;  // link to parent node
  LinkedList<T> _children;  // links to children
  T _data;  // data element the node stores
}
```

Node class for binary trees

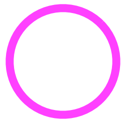
- From *binary* trees, we can define a `Node` more simply:

```
class Node<T> {
  Node<T> _parent;
  Node<T> _leftChild, _rightChild;
  T _data;
}
```

Defined to be `null` if child does not exist.

- We can then begin creating `Nodes` and assembling a tree:

```
final Node<String> root = new Node<String>();
root._leftChild = new Node<String>();
root._rightChild = new Node<String>();
root._rightChild._leftChild = new Node<String>();
```



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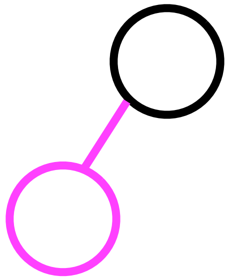

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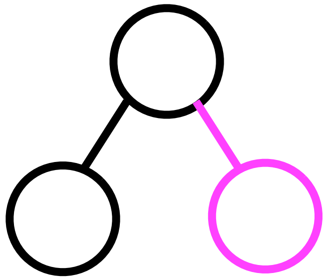

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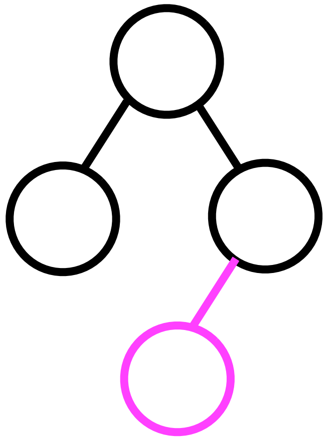

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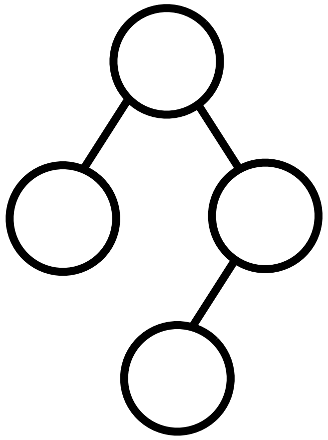

Tree operations

- We will consider three fundamental operations:
  - **`add (o, parent, leftOrRight)`** -- add a new node (containing the object o) as the leftOrRight child of the specified parent.
  - **`find (o)`** -- find and return the node containing data **`o`**.
  - **`remove (o)`** -- remove the node containing the specified data.
- Note that these operations will be used *internally* by ADTs we develop *based on* trees.
  - This is why we find and return the *node* instead of the data contained *inside* the node.
  - They will *not* be exposed to the user of, say, the Heap ADT, which is built using a binary tree.

Adding a node

- Given the parent node, it is straightforward to add a new node that is either the left or right child of the parent:

```
void add (T o, Node<T> parent,
          ChildType leftOrRight) {
  final Node<T> node = new Node<T>();
  node._data = o;
  if (leftOrRight == ChildType.LEFT_CHILD) {
    parent._leftChild = node;
  } else {
    parent._rightChild = node;
  }
}
```

Adding a node

- Given the parent node, it is straightforward to add a new node that is either the left or right child of the parent:

A Java *enumeration* type.

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  if (leftOrRight == ChildType.LEFT_CHILD) {
    parent._leftChild = node;
  } else {
    parent._rightChild = node;
  }
}
```

Java enumerations

- Enumerations are types that contain only a few possible values.
- Each value in the enumeration can be given a meaningful name,.
- If we define an enumeration type called **`ChildType`**:

```
enum ChildType {
  LEFT_CHILD, RIGHT_CHILD
}
```

- ...then we can declare and use a variable of that type:

```
ChildType leftOrRight = ChildType.RIGHT_CHILD;
```

Java enumerations

- Instead of defining an enumeration type, one could instead just use an integer and "assign" meaning to these values, e.g.:

```
int leftOrRight;
leftOrRight = 1;  // 1 indicates left child
leftOrRight = 2;  // 2 indicates right child
...
if (leftOrRight == 2) {
  // Do something with the right child
}
```

- But what if `leftOrRight` was somehow set to an invalid value?
  - With enumerations, the Java compiler *prevents* this possibility from ever happening.

```
ChildType leftOrRight = 3;  // Won't compile
```

Finding a node

- Finding a node in a binary tree is best implemented using recursion. We'll let `node` represent the root of the *sub-tree* we are currently searching.

```
Node<T> find (Node<T> root, T o) {
  if (root._data.equals(o)) {
    return root;
  }
  Node<T> node;
  if (_leftChild != null &&
      (node = find(_leftChild, o)) != null) {
    return node;
  } else if (_rightChild != null &&
      (node = find(_rightChild, o)) != null) {
    return node;
  } else {
    return null;
  }
}
```

Combined assignment to `node` and comparison to null. This is compact notation, but it sometimes can also yield more readable code.

Finding a node

- Watch how the method works for `find(a, “e”)`:

root: a

```
Node<T> find (Node<T> root, T o) {
  if (root._data.equals(o)) {
    return root;
  }
  Node<T> node;
  if (_leftChild != null &&
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    return node;
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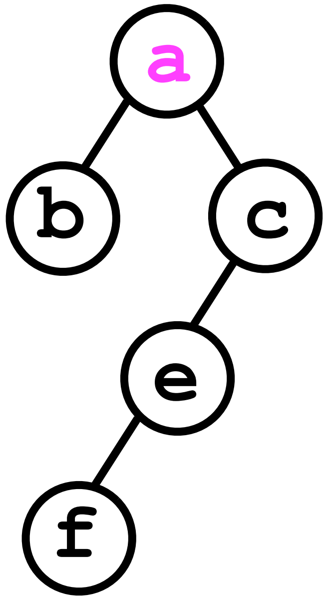
      (node = find(_rightChild, o)) != null) {
    return node;
  } else {
    return null;
  }
}
```

No



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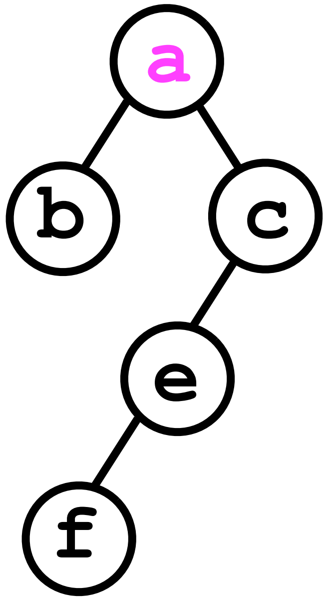

Finding a node

- Watch how the method works for `find(a, "e")`:

root: b

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Node<T> find (Node<T> root, T o) {
  if (root._data.equals(o)) {
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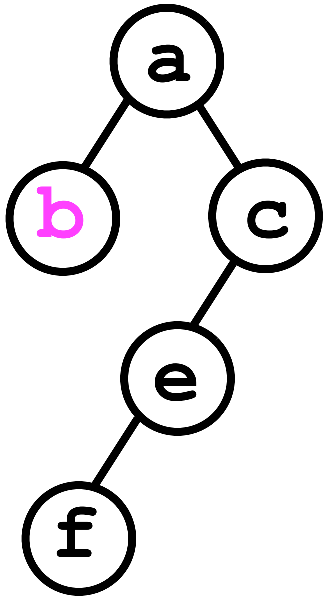

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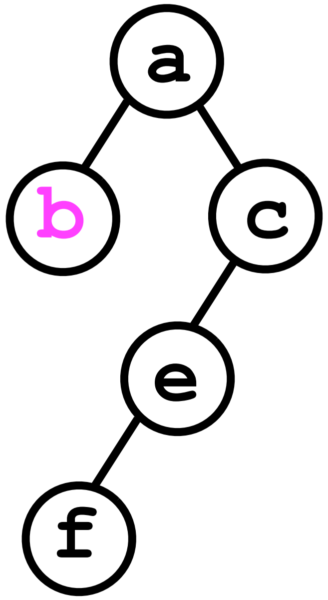

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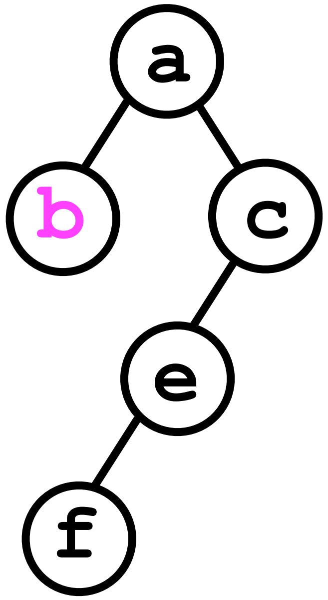

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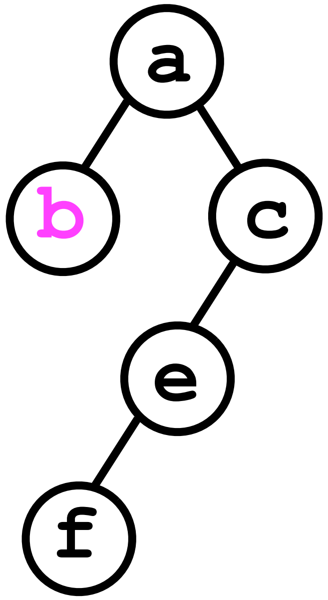

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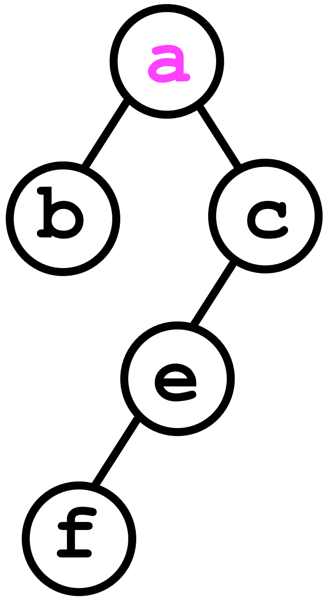

Finding a node

- Watch how the method works for `find(a, "e")`:

root: c

```
Node<T> find (Node<T> root, T o) {
  if (root._data.equals(o)) {   No
    return root;
  }
  Node<T> node;
  if (_leftChild != null &&
      (node = find(_leftChild, o)) != null) {
    return node;
  } else if (_rightChild != null &&
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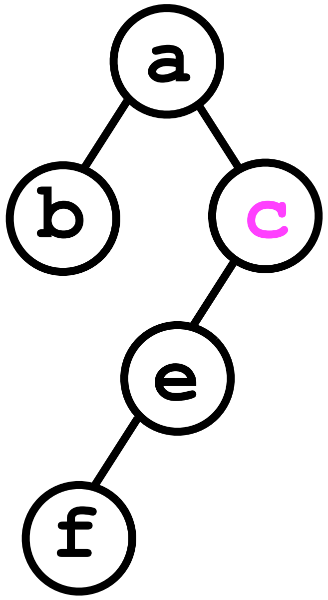
}
```



Finding a node

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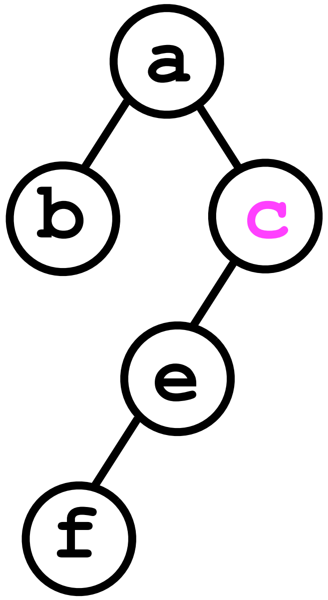

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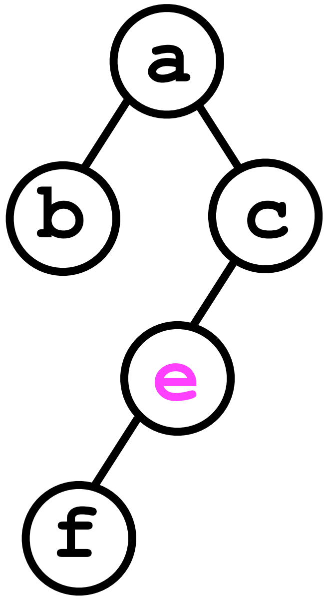
YES!!!!!!!!!!!!!!!!!!!!!!!!!!!!!



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```
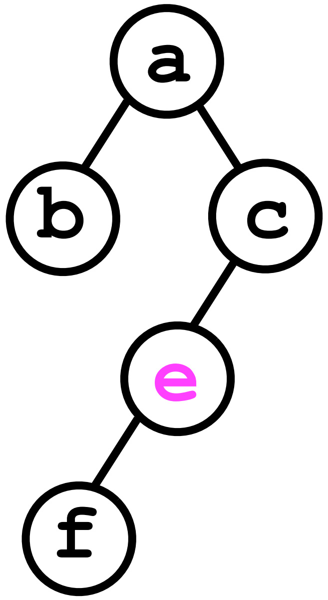

The returned node will "propagate back up" the recursive calls.



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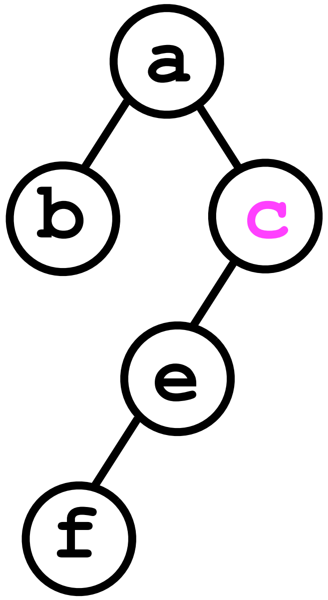
  }
  Node<T> node;
  if (_leftChild != null &&
      (node = find(_leftChild, o)) != null) {
    return node;
  } else if (_rightChild != null &&
      (node = find(_rightChild, o)) != null) {
    return node;
  } else {
    return null;
  }
}
```



Finding a node

- Watch how the method works for `find(a, “e”)`:

root: a

```
Node<T> find (Node<T> root, T o) {
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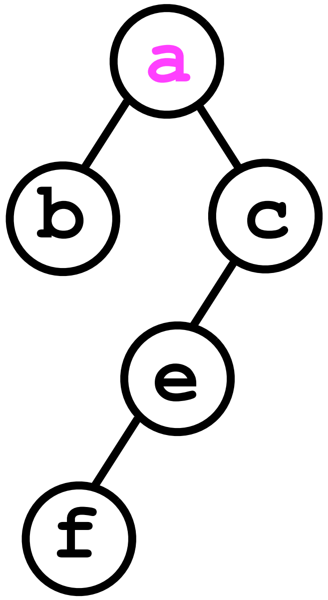

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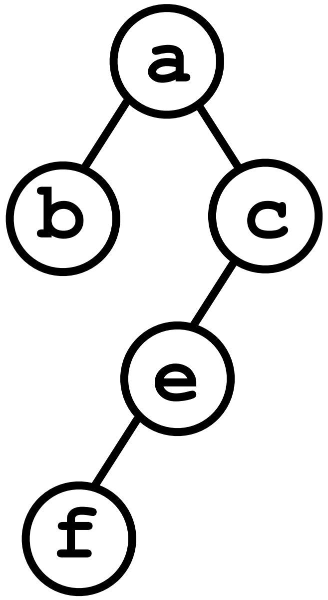
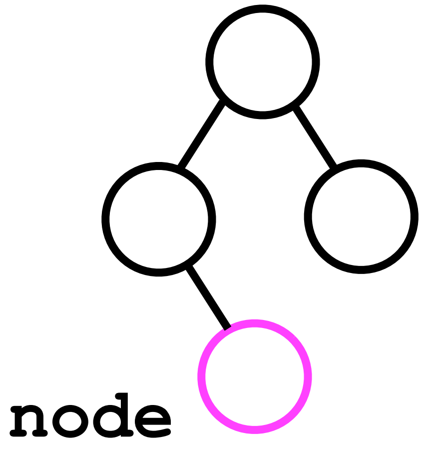


Done!!!!!!!!!!!!!!!!!!!!
!!!!!!!!!!!!!!!!!!!!!!

Removing a node

- How to implement the `remove(o)` operation depends on whether the node containing o is a *leaf* node or an *internal* node.
  - We can use the find method to locate the correct node.
- If the node is a leaf, then we just "snip" it off from its parent, e.g.:

```
node._parent._rightChild = null;
```



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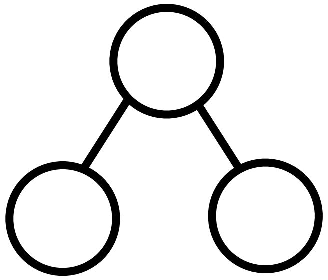
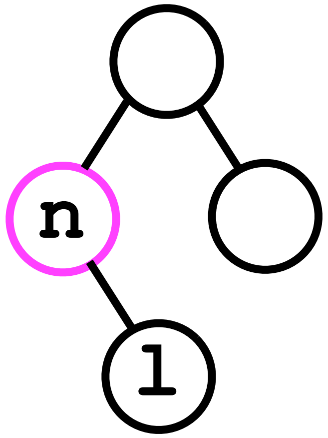


Removing a node

- If, however, the node is an internal node, then "snipping" it off would remove the whole sub-tree.

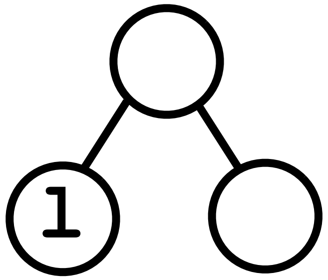
- To just remove the node but not its children, we need to *replace* the internal node with some other node.

- Instead of actually removing and replacing `n`, we can instead just replace the *data* it stores with the data of another *leaf* node (e.g., `l`).
  - We can then remove the "old" `l`.



Removing a node

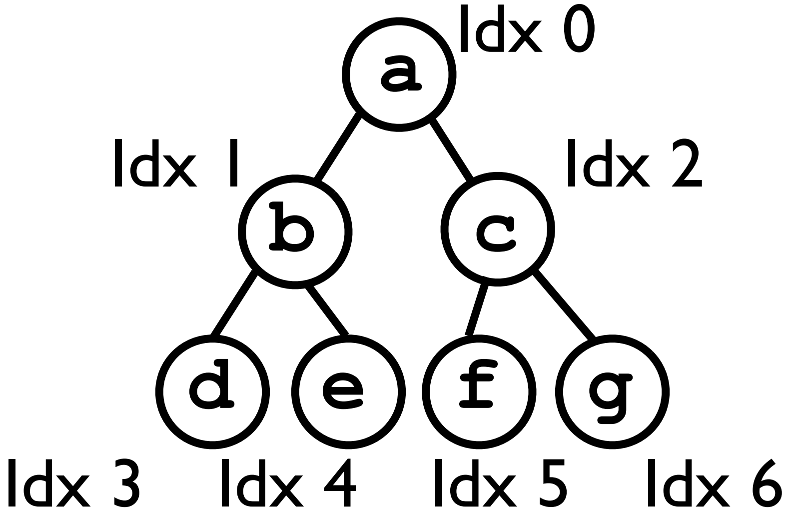
- If, however, the node is an internal node, then "snipping" it off would remove the whole sub-tree.
- To just remove the node but not its children, we need to *replace* the internal node with some other node.
- Instead of actually removing and replacing **`n`**, we can instead just replace the *data* it stores with the data of another *leaf* node (e.g., `1`).
  - We can then remove the "old" `1`.



Array-based binary trees.

Array-based binary trees

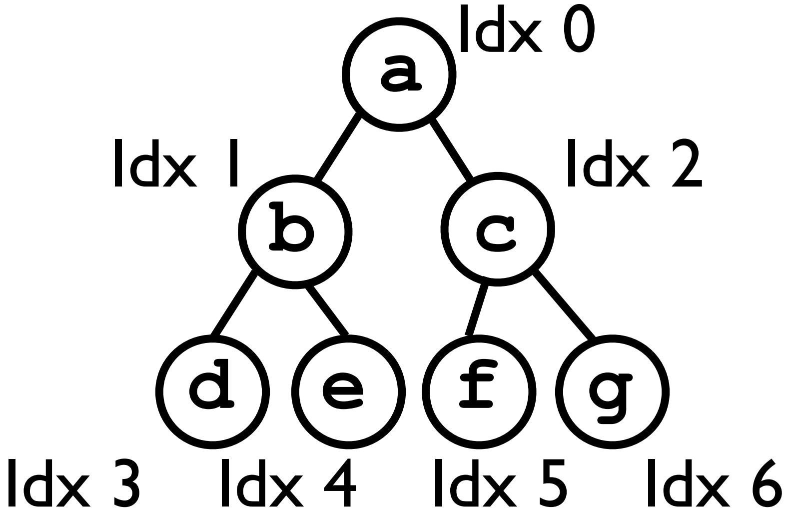
- Just as *lists* can be implemented by either a linked chain of `Nodes` or an array, a *binary tree* can be implemented as a tree of `Nodes` or an array as well.
- Each "node" in the tree will be assigned a unique index at which its *data* should be stored.
- Given the index of a particular "node", the index of its *parent*, and the indices of its *children*, can be easily computed.



a	b	c	d	e	f	g
0	1	2	3	4	5	6

Array-based binary trees

- The index(*n*) of a node *n* with parent *p* is:
    - 0 if *n* is the root node.
    - 2*index(*p*)+1 if *n* is left child of *p*.
    - 2*index(p)+2 if n is right child.
- The parentIndex(`idx`) of a node stored at `idx` is `(idx-1)`/2.
- Examples:
  index(c) = 2*index(a)+2 = 2*0+2 =1
  parentIndex(4) = (4-1)/2 = 1.5 = 1.



a	b	c	d	e	f	g
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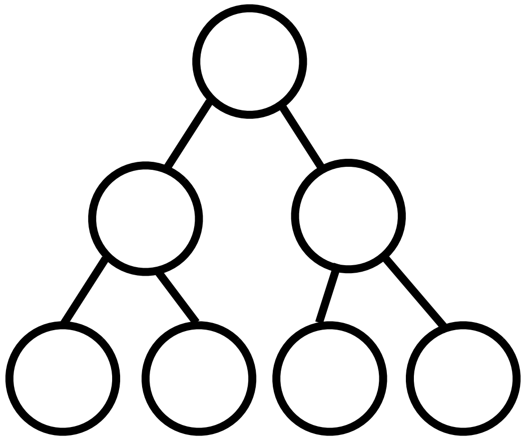
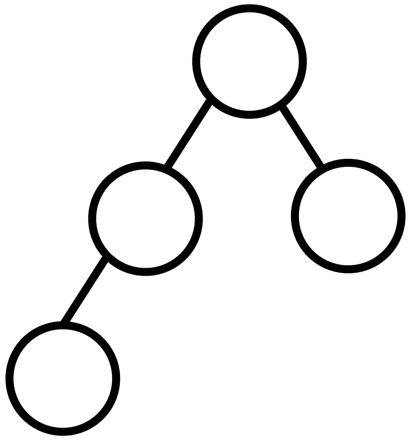
Array-based binary trees

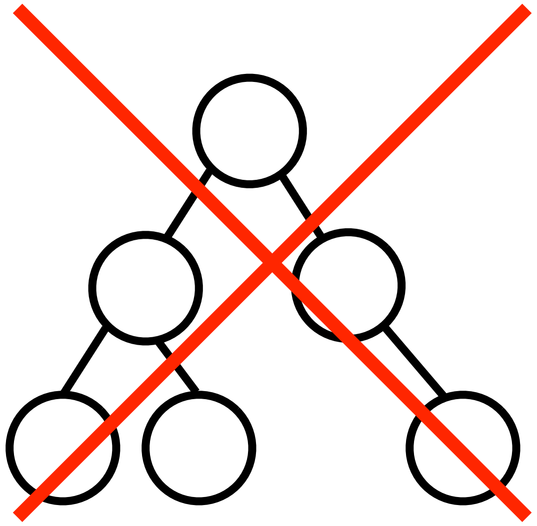
- Note that this array-based representation applies only to *complete* binary trees.
  - A binary tree is *complete* if every level of the tree is completely filled except possibly the last *and* the last level is (partially) filled from left to right.



OK



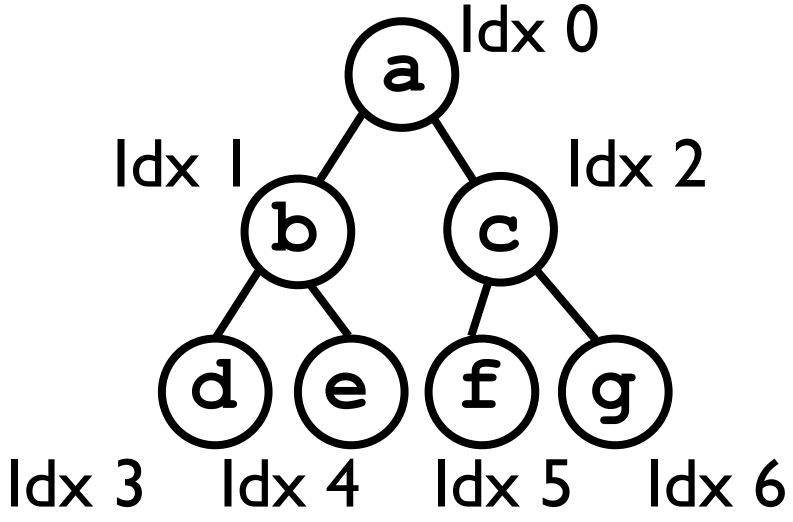
OK



Not OK

Array-based binary trees

- Even though the data are being stored in a regular Java array, *their locations in the array still encode a tree structure among them*.
    - This means that binary tree-based algorithms we develop can still offer time-cost advantages over linear lists.

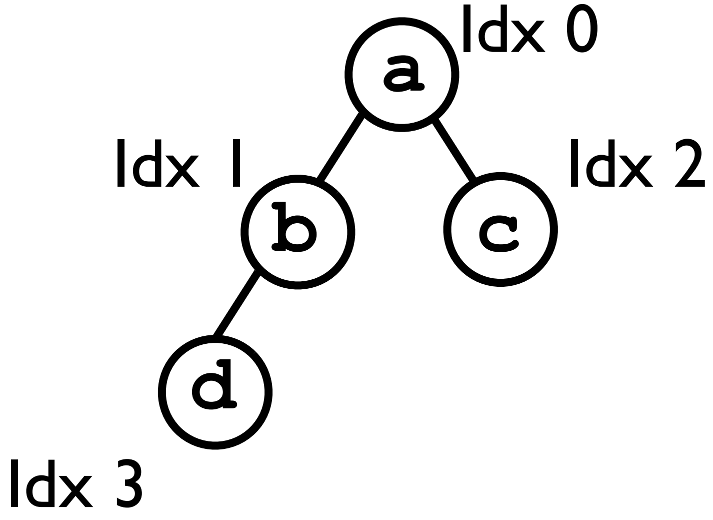


`a`	`b`	`c`	`d`	`e`	`f`	`g`
0	1	2	3	4	5	6

Adding a node

- Given that the binary tree must be *complete*, it is only valid to add a node *n* to be the *next child on the last level of the tree.*

- The index into the array of where this “next child” should be stored is always just **`_numNodes`**, where **`_numNodes`** is the current number of nodes in the tree.



a	**b**	**c**	**d**			
0	1	2	3	4	5	6

`_numNodes: 4`

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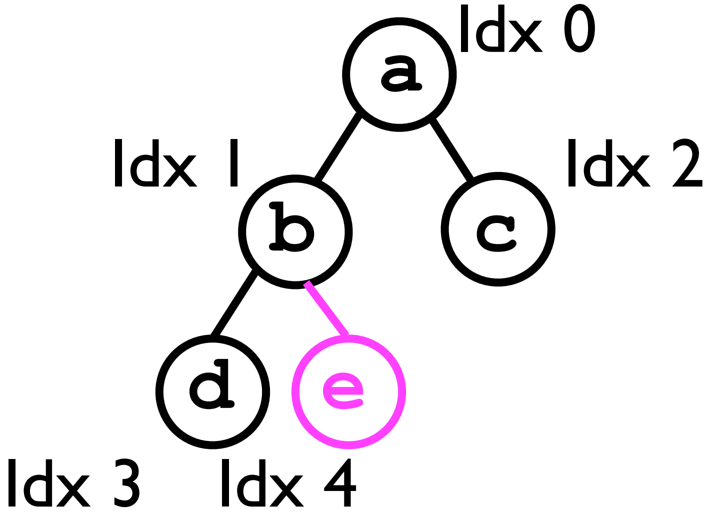
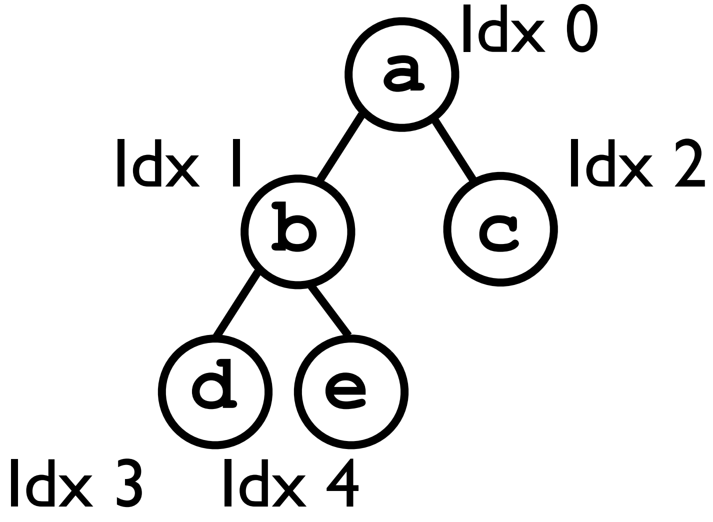


a	b	c	d	e		
0	1	2	3	4	5	6

`_numNodes: 5`

Removing a node

- Similarly, it is only valid to remove the right-most child of the last level of the tree.
- All we must do is decrement **`_numNodes`** to indicate that the “slot” in the array of the removed node is no longer valid.



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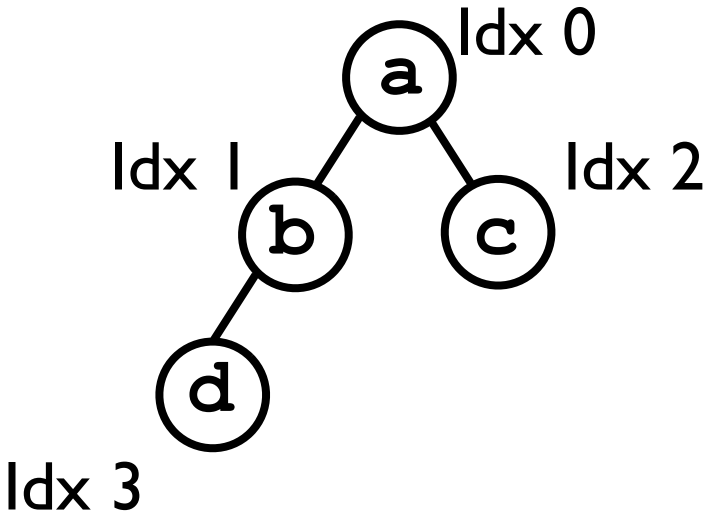


a	b	c	d			
0	1	2	3	4	5	6

`_numNodes: 4`

Finding a node

- To find the index of a node *n* whose data element equals `o`:

```
int find (int rootIdx, T o) {
  if (_nodeArray[rootIdx].equals(o)) {
    return rootIdx;
  }

  int idx;
  if (leftChild(rootIdx) < _numNodes &&
      (idx = find(leftChild(rootIdx), o)) >= 0) {
    return idx;
  } else if (rightChild(rootIdx) < _numNodes &&
      (idx = find(rightChild(rootIdx), o)) >= 0) {
    return idx;
  } else {
    return -1;
  }
}
```

Make sure each child exists before recursing!

Helper methods to determine index of left and right child nodes.

Binary trees to accelerate search.

Binary trees to accelerate search

- We have now constructed considerable "infrastructure" for building binary trees, using either "linked nodes" or a Java array for the tree's underlying storage.
- Trees are useful in their own right for representing *hierarchical structures*, e.g., genealogical data.
- However, for the moment we are interested in how they can *store* and *accelerate search* of data on which an *ordering relation* is defined.

Binary trees to accelerate search

- *Heaps* and *binary search trees* are two ADTs based on binary trees that accelerate search.
- A heap offers fast access to the largest element in a collection of related objects.
  - $O(1)$ worst-case time cost for findLargest.
  - $O(\log n)$ worst-case time cost for removeLargest.
  - $O(\log n)$ worst-case time cost for add.
  - $O(n)$ worst-case time-cost for find and remove.

Binary trees to accelerate search

- A binary search tree (BST) offers:
  - *O*(log *n*) average-case time cost for add, find, remove, and findLargest.
  - *O*(*n*) worst-case time cost for add, find, remove, and findLargest.
- AVL trees and red-black trees are more complicated, but they offer:
  - *O*(log *n*) worst-case time cost for add, find, remove, and findLargest.

Why findLargest?

- Why would we want to find the largest data element stored in a container?
- The `findLargest` method is required by *priority queues.*
    - A *priority queue* is a queue in which elements are dequeued not in FIFO order, but instead *in order of highest-to-lowest priority*.
    - A priority queue is typically implemented using a *heap*.

Highest priority person

Taken from Paul Kube's slides.

Heaps.

Heaps

- A *max-heap* is an ADT for storing data so that the *largest element* (according to some binary order relation) can always be found and removed quickly.
- A *min-heap* is defined analogously for the *smallest element*.
- Internally, a *heap* is a *complete* binary tree which satisfies the *heap condition*:
  - The root of every sub-tree is *no smaller than any node in the sub-tree*. (For *max*-heap).
  - The root of every sub-tree is *no larger than any node in the sub-tree*. (For *min*-heap).
- This ensures that, to implement findLargest/findSmallest, *we can always just return the root node of the tree*.

Heaps

- A *max-heap* has the following interface:

```
// All operations must preserve the heap condition.
interface MaxHeap {
  // Adds o to the heap.
  void add (T o);
  // Removes the node whose data element equals o.
  void remove (T o);
  // Removes and returns the largest node in the heap.
  T removeLargest ();
  // Returns the largest node in the heap.
  T findLargest ();
  // Finds and returns the node whose data element
  // equals o.
  T find (T o);
  // Returns the number of data stored in the heap.
  int size ();
}
```

Implementing heaps

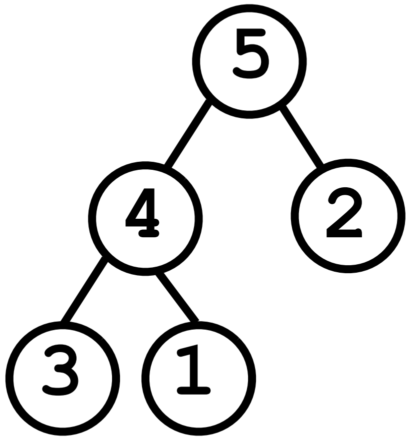
- Since heaps are anyway a *complete* binary tree, it is convenient and efficient to implement them using an array.
    - However, they could also be implemented using linked nodes.
- The challenge when implementing a heap is to preserve the heap property upon every *mutation* to the heap (add/remove).

Adding a node to a heap

- In order to add a new element o to a max-heap while *preserving the heap condition*, we execute the following procedure:
    - Add a new node *n* containing o to the last level of the tree (ensure *completeness* of the tree).
        - *This may violate the tree's heap condition* because o may be larger than one of its parents.
    - We then "fix" the heap by "swapping" node *n* with its parent *p*.
        - We repeat this process -- known as *bubbling* up -- as many times as necessary until the tree is a *heap* again.

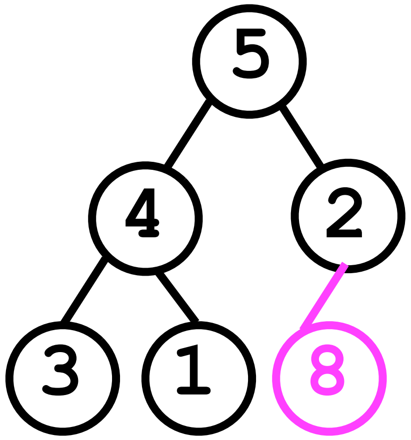
Adding a node to a heap

- Consider the heap to the right. (Notice that it satisfies the *heap condition*).



Adding a node to a heap

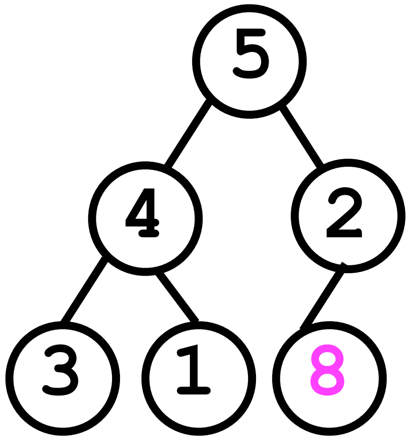
- Consider the heap to the right. (Notice that it satisfies the *heap condition*).
- Suppose we add value 8 to the bottom-level of the heap.
  - The tree no longer satisfies the heap condition.



2 is smaller than one of the nodes in its sub-tree!

Adding a node to a heap

- Consider the heap to the right. (Notice that it satisfies the *heap condition*).
- Suppose we add value 8 to the bottom-level of the heap.
  - The tree no longer satisfies the heap condition.
  - We have to "bubble up" the 8 we just added to restore the heap condition.



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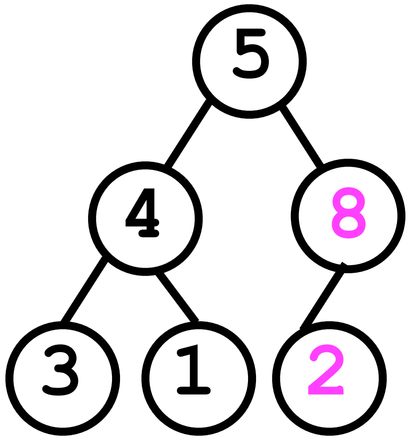

Not done yet -- 5 is still smaller than 8.

Adding a node to a heap

- Consider the heap to the right. (Notice that it satisfies the *heap condition*).
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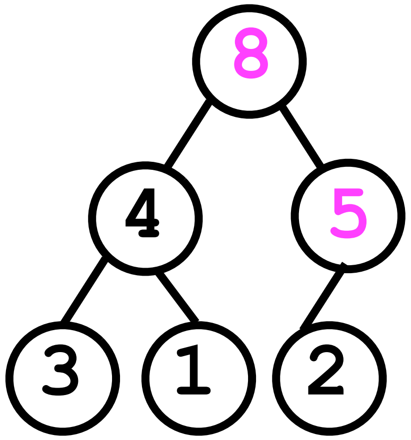
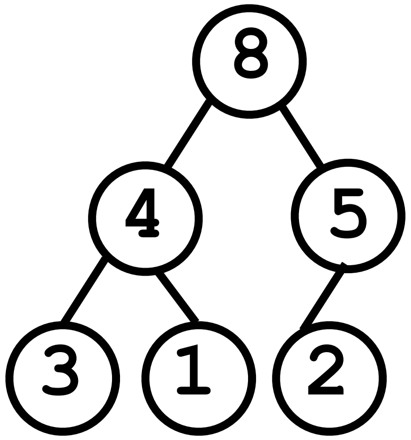


Now it is a heap again!

Adding a node to a heap



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- Suppose we add value 8 to the bottom-level of the heap.
    - The tree no longer satisfies the heap condition.
    - We have to "bubble up" the 8 we just added to restore the heap condition.
    - Done!

Adding a node to a heap

- We can implement the `add(o)` method as:

```
void add (T o) {
  _nodeArray[_numNodes] = o;
  _numNodes++;
  bubbleUp(_numNodes - 1);
}
```

- We must then also implement `bubbleUp(idx)`:

```
void bubbleUp (int idx) {
  If node at idx is “larger” than its parent:
    Swap data in the node and its parent;
    Recursively bubbleUp(parentIdx(idx));
}
```

Adding a node to a heap

- Alternatively, we can write an *iterative* version of `bubbleUp(idx)`:

```
void bubbleUp (int idx) {
  While node at idx is “larger” than its parent:
    Swap data in the node and its parent;
    Set idx to be parentIdx(idx);
}
```

Next lecture

- Finding and removing elements.
- “Trickling down” a node.